

SURFACES Ω AND THEIR TRANSFORMATIONS*

BY

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INTRODUCTION

In a recent memoir† we showed that a surface S referred to a conjugate system with equal point invariants admits of transformations into surfaces upon which the parametric curves are of the same kind such that, if S_1 is a transform, for the congruence (G) of lines joining corresponding points on S and S_1 the developables cut these surfaces in the parametric curves, and the focal points on any line of the congruence are harmonic with respect to the points of S and S_1 on the line. In this case we say that S and S_1 are in the relation of a *transformation* K . In the present memoir we consider the particular case when the congruence (G) is a normal one. Surfaces orthogonal to such a congruence have been considered by Demoulin‡ who called them *surfaces* Ω . He gives practically no consideration to the surfaces S and S_1 which are fundamental in the discussion of the present paper. They belong to a particular class of surfaces, which we call *surfaces* C . It is by the characterization of surfaces C and the establishment of transformations K of them that we arrive at a general theory of transformations of surfaces Ω .

As shown in § 2 surfaces C are characterized by the property that the parametric conjugate system with equal point invariants is 2, O in the sense of Guichard,§ the complementary function being the distance to a surface Ω normal to the congruence (G). From the general theory of systems 2, O it follows that the spheres whose centers lie on a surface C and whose radii are equal to the complementary function envelop two surfaces upon which the lines of curvature correspond, and these surfaces are said to be in the relation of a transformation of Ribaucour. Hence there are two congruences (G) associated with a surface C , and with each congruence another surface C , namely the surface S_1 referred to above, but which hereafter will be called the surfaces C_0 and C'_0 . We say that C_0 and C'_0 are the conjugate surfaces of C .

* Presented to the Society, January 1, 1915.

† These Transactions, vol. 15 (1914), pp. 397-430. A reference to an equation of this memoir will be of the form M. (—).

‡ Comptes Rendus, vol. 153 (1911), pp. 590-593, 705-707, 927-929.

§ Annales de l'école normale supérieure, ser. 3, vol. 14 (1897), pp. 467-516.

The surface C and each of these surfaces are in the relation of a transformation K such that the congruence (G) is normal.

In § 6 we establish the existence of transformations K_m of surfaces C into surfaces of the same kind, the subscript m denoting an essential constant which enters in the equations of the transformation. We use the subscript only in connection with transformations K which transform a surface C into a surface C . Each of these transformations gives rise to a transformation of the Ribaucour type of surfaces Ω , which we call a *transformation* A_m .

In a previous paper we showed that transformations K admit a theorem of permutability. We generalize it in the present memoir so as to give eight surfaces interrelated by transformations K . By means of this result we establish a theorem of permutability for the transformations K_m of surfaces C and for the transformations A_m of surfaces Ω .

Demoulin* showed that with a given surface Ω there is associated a second surface Ω such that the two surfaces have the same spherical representation of their lines of curvature. It can be shown that this relation is a special case of the transformations A_m .

An isothermic surface is a surface C , but in this case the congruence (G) is normal to C and hence C coincides with the surface Ω determined by it. Now the transformations A_m are the well-known transformations D_m discovered by Darboux and investigated at length by Bianchi.†

In the closing section we consider the case where a surface C is the middle surface of the congruence (G) , under which condition the surface C_0 is at infinity. Now the surfaces normal to (G) are surfaces with isothermic spherical representation of their lines of curvature. We have studied‡ the transformations of these surfaces in a former memoir and now join the earlier results with those of the present study.

When the Lie line-sphere transformation is applied to a surface Ω , the resulting surface possesses a conjugate system whose tangents form W -congruences. Surfaces of this sort have been studied by Tzitzéica,§ who called them *surfaces* R . A part of Demoulin's investigations have to do with surfaces R . We content ourselves here with the remark that the results of the present memoir establish the existence of transformations of surfaces R , such that a surface and its transform are the focal surfaces of a W -congruence.||

In a subsequent memoir we shall put the equations of a transformation A_m in a different form and study certain types of surfaces Ω .

* Loc. cit., p. 928.

† For a full discussion of this case, see M., pp. 422-428.

‡ These Transactions, vol. 9 (1908), pp. 149-177.

§ Comptes Rendus, vol. 153 (1911), p. 1127.

|| Cf. Tzitzéica, loc. cit.

1. EQUATIONS OF GENERAL TRANSFORMATIONS K

If S is a surface referred to a conjugate system with equal point invariants, its cartesian coördinates x, y, z , are solutions of an equation of the form

$$(1) \quad \frac{\partial^2 \theta}{\partial u \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v} \frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u} \frac{\partial \theta}{\partial v} = 0,$$

where in general ρ is a function of u and v . If θ_1 is solution of this equation, linearly independent of x, y , and z , and λ_1 is the function defined by

$$(2) \quad \frac{\partial \lambda_1}{\partial u} = -\rho \frac{\partial \theta_1}{\partial u}, \quad \frac{\partial \lambda_1}{\partial v} = \rho \frac{\partial \theta_1}{\partial v},$$

the functions x_1, y_1, z_1 , given by equations of the form

$$(3) \quad \begin{aligned} \frac{\partial x_1}{\partial u} &= \frac{\rho}{\lambda_1} \left[(x_1 - x) \frac{\partial \theta_1}{\partial u} + \theta_1 \frac{\partial x}{\partial u} \right], \\ \frac{\partial x_1}{\partial v} &= -\frac{\rho}{\lambda_1} \left[(x_1 - x) \frac{\partial \theta_1}{\partial v} + \theta_1 \frac{\partial x}{\partial v} \right], \end{aligned}$$

are solutions of the equation

$$(4) \quad \frac{\partial^2 \theta'}{\partial u \partial v} + \frac{\partial \log \sqrt{\rho_1}}{\partial v} \frac{\partial \theta'}{\partial u} + \frac{\partial \log \sqrt{\rho_1}}{\partial u} \frac{\partial \theta'}{\partial v} = 0,$$

where

$$(5) \quad \sqrt{\rho_1} = \lambda_1 / \sqrt{\rho} \theta_1.$$

Evidently the parametric curves on S_1 , whose cartesian coördinates are x_1, y_1 , and z_1 , form a conjugate system with equal point invariants. Moreover, we have shown* that for the congruence (G) of lines joining corresponding points on S and S_1 the latter points are harmonic with respect to the focal points. The coördinates of the focal points are of the form

$$(6) \quad \frac{\lambda_1 x_1 - \rho \theta_1 x}{\lambda_1 - \rho \theta_1}, \quad \frac{\lambda_1 x_1 + \rho \theta_1 x}{\lambda_1 + \rho \theta_1}.$$

If 2ω denotes the angle between the coördinate curves at a point of S , and if X_1, Y_1, Z_1 and X_2, Y_2, Z_2 denote the direction-cosines of the bisectors of the angles between the tangents to these curves at the point, we have

$$(7) \quad \frac{\partial x}{\partial u} = \sqrt{E} (\cos \omega X_1 - \sin \omega X_2), \quad \frac{\partial x}{\partial v} = \sqrt{G} (\cos \omega X_1 + \sin \omega X_2),$$

where E, F , and G are the fundamental coefficients of S and

$$(8) \quad F = \sqrt{EG} \cos 2\omega.$$

* M., p. 400.

We introduce three functions a_1, b_1, w_1 by the equation

$$(9) \quad x_1 - x = \frac{1}{m_1 \lambda_1} (a_1 X_1 + b_1 X_2 + w_1 X),$$

where m_1 is an arbitrary constant, it being understood that similar equations hold for y_1 and z_1 with the same functions a_1, b_1, w_1 .

When the above expression for x_1 is substituted in equations (3), we find that a_1, b_1, w_1 must satisfy the equations

$$\begin{aligned} \frac{\partial a_1}{\partial u} &= -m_1(\lambda_1 - \rho\theta_1) \sqrt{E} \cos \omega + b_1 A + w_1 D / (2 \sqrt{E} \cos \omega), \\ \frac{\partial a_1}{\partial v} &= -m_1(\lambda_1 + \rho\theta_1) \sqrt{G} \cos \omega - b_1 B + w_1 D'' / (2 \sqrt{G} \cos \omega), \\ (10) \quad \frac{\partial b_1}{\partial u} &= m_1(\lambda_1 - \rho\theta_1) \sqrt{E} \sin \omega - a_1 A - w_1 D / (2 \sqrt{E} \sin \omega), \\ \frac{\partial b_1}{\partial v} &= -m_1(\lambda_1 + \rho\theta_1) \sqrt{G} \sin \omega + a_1 B + w_1 D'' / (2 \sqrt{G} \sin \omega), \\ \frac{\partial w_1}{\partial u} &= -\frac{D}{2 \sqrt{E}} \left(\frac{a_1}{\cos \omega} - \frac{b_1}{\sin \omega} \right), \quad \frac{\partial w_1}{\partial v} = -\frac{D''}{2 \sqrt{G}} \left(\frac{a_1}{\cos \omega} + \frac{b_1}{\sin \omega} \right), \end{aligned}$$

where

$$(11) \quad A = \frac{\sqrt{E}}{\sqrt{G}} \frac{\partial \log \sqrt{\rho}}{\partial v} \sin 2\omega - \frac{\partial \omega}{\partial u}, \quad B = \frac{\sqrt{G}}{\sqrt{E}} \frac{\partial \log \sqrt{\rho}}{\partial u} \sin 2\omega - \frac{\partial \omega}{\partial v}.$$

If we put for the sake of brevity

$$(12) \quad T_1^2 = a_1^2 + b_1^2 + w_1^2,$$

it follows from (10) that

$$\begin{aligned} T_1 \frac{\partial T_1}{\partial u} &= m_1(\rho\theta_1 - \lambda_1) \sqrt{E} (a_1 \cos \omega - b_1 \sin \omega), \\ (13) \quad T_1 \frac{\partial T_1}{\partial v} &= -m_1(\rho\theta_1 + \lambda_1) \sqrt{G} (a_1 \cos \omega + b_1 \sin \omega). \end{aligned}$$

2. CHARACTERIZATION OF SURFACES Ω

In the preceding section we denoted by (G) the congruence formed by the lines joining corresponding points on a surface S and on a surface arising from S by a transformation K . We consider now the case when such a congruence (G) is a normal. We shall find that the surface S is not a general one. Any surface which leads to such a normal congruence we call a *surface C* . We assume that such a surface is known and we denote by C_0 the conjugate surface which with C determines the congruence. We indicate by a subscript

zero functions belonging to C_0 and also the functions which define the transformation K from C into C_0 . Furthermore, we say that C_0 is in the relation of a transformation K_0 to C .

From (9) and (12) it follows that the distance between corresponding points M and M_0 on C and C_0 is $T_0/\lambda_0 m_0$. Hence the coördinates \bar{x} , \bar{y} , \bar{z} , of a point on the line MM_0 are of the form

$$(14) \quad \bar{x} = x + \frac{t \lambda_0 m_0}{T_0} (x_0 - x).$$

In order that this point describe a surface \bar{S} normal to the lines of the congruence (G) , it is necessary and sufficient that

$$\sum (x_0 - x) \frac{\partial \bar{x}}{\partial u} = 0, \quad \sum (x_0 - x) \frac{\partial \bar{x}}{\partial v} = 0.$$

In consequence of equations (13) these conditions lead to

$$(15) \quad \begin{aligned} \frac{\partial t}{\partial u} + \frac{\sqrt{E}}{T_0} (\cos \omega a_0 - \sin \omega b_0) &= 0, \\ \frac{\partial t}{\partial v} + \frac{\sqrt{G}}{T_0} (\cos \omega a_0 + \sin \omega b_0) &= 0. \end{aligned}$$

If these equations be differentiated with respect to v and u respectively, we have

$$(16) \quad C + (\rho \theta_0 + \lambda_0) D = 0, \quad C - (\rho \theta_0 - \lambda_0) D = 0,$$

where we have put for the sake of brevity

$$\begin{aligned} C &= \frac{\partial^2 t}{\partial u \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v} \frac{\partial t}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u} \frac{\partial t}{\partial v}, \\ D &= m_0 \sqrt{EG} (a_0^2 \cos^2 \omega - b_0^2 \sin^2 \omega - T_0^2 \cos 2\omega T_0^3). \end{aligned}$$

From the form of (16) it follows that C and D must be zero. The first shows that t is a solution of equation (1) and the second necessitates

$$(17) \quad a_0^2 \cos^2 \omega - b_0^2 \sin^2 \omega - T_0^2 \cos 2\omega = 0.$$

In consequence of (12) this equation may be written in the form

$$(17') \quad (b_0^2 + w_0^2) \cos^2 \omega - (a_0^2 + w_0^2) \sin^2 \omega = 0.$$

From equations (8), (15), and (17) it follows that

$$(18) \quad \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial t}{\partial u} \frac{\partial t}{\partial v} = 0.$$

Hence $x^2 + y^2 + z^2 - t^2$ is a solution of equation (1) as well as t . Adopting the notation of Guichard,* we have that the parametric conjugate system on C is $2, O$ the complementary function being t .

It is a well-known fact† that when a conjugate system $2, O$ is known on any surface Σ , the spheres whose centers lie on Σ and whose radii are equal to the complementary function are enveloped by two surfaces, upon which the lines of curvature correspond to the given conjugate system on Σ . Moreover, this is a characteristic property of envelopes of spheres with lines of curvature in correspondence. It is our purpose to show that when the system $2, O$ has equal point invariants the two envelopes are surfaces Ω .

Let $\bar{x}, \bar{y}, \bar{z}$ denote the cartesian coördinates of a surface \bar{S} referred to its lines of curvature, $\bar{X}, \bar{Y}, \bar{Z}$ the direction-cosines of its normal, ρ_1, ρ_2 its principal radii of curvature. The point whose cartesian coördinates x, y, z are defined by

$$(19) \quad x = \bar{x} - \frac{\alpha}{\beta} \bar{X}, \quad y = \bar{y} - \frac{\alpha}{\beta} \bar{Y}, \quad z = \bar{z} - \frac{\alpha}{\beta} \bar{Z},$$

describes a surface S upon which the parametric system is conjugate and $2, O$, provided that α and β are functions of u and v satisfying the equations§

$$(20) \quad \frac{\partial \alpha}{\partial u} + \rho_1 \frac{\partial \beta}{\partial u} = 0, \quad \frac{\partial \alpha}{\partial v} + \rho_2 \frac{\partial \beta}{\partial v} = 0.$$

In this case \bar{S} is one of the sheets of the envelope of the spheres with centers on S and of radius α/β .

The functions for \bar{S} satisfy

$$(21) \quad \frac{\partial \bar{x}}{\partial u} + \rho_1 \frac{\partial \bar{X}}{\partial u} = 0, \quad \frac{\partial \bar{x}}{\partial v} + \rho_2 \frac{\partial \bar{X}}{\partial v} = 0,$$

and similar equations in \bar{y} and \bar{z} . Moreover, \bar{x}, \bar{y} , and \bar{z} are solutions of the equation

$$(22) \quad \frac{\partial^2 \bar{\theta}}{\partial u \partial v} = \frac{\partial}{\partial v} \log \sqrt{\bar{E}} \frac{\partial \bar{\theta}}{\partial u} + \frac{\partial}{\partial u} \log \sqrt{\bar{G}} \frac{\partial \bar{\theta}}{\partial v},$$

where \bar{E} and \bar{G} are the first fundamental coefficients of \bar{S} .

With the aid of (21) we obtain from (19) by differentiation

$$(23) \quad \begin{aligned} \frac{\partial x}{\partial u} &= \left(1 + \frac{\alpha}{\beta \rho_1}\right) \left(\frac{\partial \bar{x}}{\partial u} - \frac{1}{\beta} \frac{\partial \alpha}{\partial u} \bar{X}\right), \\ \frac{\partial x}{\partial v} &= \left(1 + \frac{\alpha}{\beta \rho_2}\right) \left(\frac{\partial \bar{x}}{\partial v} - \frac{1}{\beta} \frac{\partial \alpha}{\partial v} \bar{X}\right). \end{aligned}$$

* *Annales de l'école normale supérieure*, ser. 3, vol. 14 (1897), pp. 467-516.

† Cf. Darboux, *Leçons sur la théorie générale des surfaces*, vol. II, Paris, 1889, p. 326.

§ Cf. Darboux, l. c., p. 339.

The Codazzi equations for \bar{S} may be written*

$$(24) \quad \frac{\partial}{\partial v} \left(\frac{1}{\rho_1} \right) = \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \frac{\partial \log \sqrt{\bar{E}}}{\partial v}, \quad \frac{\partial}{\partial u} \left(\frac{1}{\rho_2} \right) = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \frac{\partial \log \sqrt{\bar{G}}}{\partial u}.$$

From (20) and (24) we have

$$(25) \quad \begin{aligned} \frac{\partial}{\partial u} \left(\frac{\alpha}{\beta \rho_2} \right) &= \left(1 + \frac{\alpha}{\beta \rho_1} \right) \frac{1}{\rho_2 \beta} \frac{\partial \alpha}{\partial u} + \frac{\alpha}{\beta} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \frac{\partial \log \sqrt{\bar{G}}}{\partial u}, \\ \frac{\partial}{\partial v} \left(\frac{\alpha}{\beta \rho_1} \right) &= \left(1 + \frac{\alpha}{\beta \rho_2} \right) \frac{1}{\beta \rho_1} \frac{\partial \alpha}{\partial v} + \frac{\alpha}{\beta} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \frac{\partial \log \sqrt{\bar{E}}}{\partial v}. \end{aligned}$$

Making use of these formulas, we obtain from (23) by differentiation and reduction

$$(26) \quad \frac{\partial^2 x}{\partial u \partial v} = \frac{\partial}{\partial v} \log \left[\sqrt{\bar{E}} \left(1 + \frac{\alpha}{\beta \rho_1} \right) \right] \frac{\partial x}{\partial u} + \frac{\partial}{\partial u} \log \left[\sqrt{\bar{G}} \left(1 + \frac{\alpha}{\beta \rho_2} \right) \right] \frac{\partial x}{\partial v}.$$

Hence a necessary and sufficient condition that S have equal point invariants is

$$(27) \quad \frac{\partial^2}{\partial u \partial v} \log \left(\frac{\sqrt{\bar{E}}}{\sqrt{\bar{G}}} \frac{1 + \alpha/\beta \rho_1}{1 + \alpha/\beta \rho_2} \right) = 0.$$

It is our purpose to show that when this condition is satisfied, the point M' harmonic to M with respect to the centers of principal curvature of \bar{S} describes a conjugate system with equal point invariants, and that consequently \bar{S} is a surface Ω .

If we put

$$(28) \quad x' = \bar{x} - \frac{\alpha'}{\beta'} X, \quad y' = \bar{y} - \frac{\alpha'}{\beta'} Y, \quad z' = \bar{z} - \frac{\alpha'}{\beta'} Z,$$

the necessary and sufficient condition that the point $M'(x', y', z')$ is the harmonic of M with respect to the centers of principal curvature of \bar{S} , the coördinates of these centers being of the form

$$(29) \quad \bar{x} + \rho_1 X, \quad \bar{x} + \rho_2 \bar{X},$$

is that α' and β' satisfy the condition

$$(30) \quad \left(\frac{\alpha}{\beta} + \rho_1 \right) \left(\frac{\alpha'}{\beta'} + \rho_2 \right) + \left(\frac{\alpha}{\beta} + \rho_2 \right) \left(\frac{\alpha'}{\beta'} + \rho_1 \right) = 0.$$

* E., pp. 157, 181. A reference of this sort is to the author's *Differential Geometry*, Ginn and Co., Boston, 1909.

In conformity with equation (30) we define a function σ by the equations

$$(31) \quad \begin{aligned} \beta' &= \sigma \left[\frac{1}{\rho_1} \left(1 + \frac{\alpha}{\beta \rho_2} \right) + \frac{1}{\rho_2} \left(1 + \frac{\alpha}{\beta \rho_1} \right) \right], \\ \alpha' &= -\sigma \left[\left(1 + \frac{\alpha}{\beta \rho_1} \right) + \left(1 + \frac{\alpha}{\beta \rho_2} \right) \right]. \end{aligned}$$

In order that the point M' shall describe a conjugate system 2, O , it is necessary and sufficient that α' and β' be a pair of solutions of equations (20). Expressing this condition, we find that σ must satisfy the equations

$$\begin{aligned} \frac{\partial \log \sigma}{\partial u} &= \frac{\alpha + \beta \rho_2}{(\rho_1 - \rho_2)(\alpha + \beta \rho_1)} \frac{\partial \rho_1}{\partial u} + \frac{\partial}{\partial u} \log \frac{\beta \rho_1 \sqrt{G}}{\alpha + \beta \rho_1}, \\ \frac{\partial \log \sigma}{\partial v} &= \frac{\alpha + \beta \rho_1}{(\rho_2 - \rho_1)(\alpha + \beta \rho_2)} \frac{\partial \rho_2}{\partial v} + \frac{\partial}{\partial v} \log \frac{\beta \rho_2 \sqrt{E}}{\alpha + \beta \rho_2}. \end{aligned}$$

With the aid of equations (24) and (25) we reduce these equations to the form

$$(32) \quad \begin{aligned} \frac{\partial \log \sigma}{\partial u} &= \frac{1}{\beta \rho_1} \frac{\partial \alpha}{\partial u} + \frac{\partial}{\partial u} \log \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) + 2 \frac{\partial}{\partial u} \log \frac{\beta \rho_1 \sqrt{G}}{\alpha + \beta \rho_1}, \\ \frac{\partial \log \sigma}{\partial v} &= \frac{1}{\beta \rho_2} \frac{\partial \alpha}{\partial v} + \frac{\partial}{\partial v} \log \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) + 2 \frac{\partial}{\partial v} \log \frac{\beta \rho_2 \sqrt{E}}{\alpha + \beta \rho_2}. \end{aligned}$$

In consequence of formulas (20) and (24) and the fact that α is a solution of equation (22) (which also is a consequence of these equations), it follows that

$$\frac{\partial}{\partial v} \left(\frac{1}{\beta \rho_1} \frac{\partial \alpha}{\partial u} \right) = \frac{\partial}{\partial u} \left(\frac{1}{\beta \rho_2} \frac{\partial \alpha}{\partial v} \right).$$

Hence the condition of integrability of equations (32) reduces to (27), and consequently the point M' , harmonic to M with respect to the centers of principal curvature of \bar{S} , describes a surface upon which the parametric curves form a conjugate system 2, O . Moreover, from (27) and (30) it follows that this conjugate system has equal point invariants.

Equation (27) may be replaced by

$$\beta + \alpha/\rho_1 = \bar{\omega} \sqrt{G} V, \quad \beta + \alpha/\rho_2 = \bar{\omega} \sqrt{E} U,$$

where $\bar{\omega}$ is a function to be determined, and U and V are functions of u and v alone respectively. Solving these equations for α and β , we obtain

$$\alpha = \frac{\bar{\omega} (\sqrt{G} V - \sqrt{E} U) \rho_1 \rho_2}{\rho_2 - \rho_1}, \quad \beta = \frac{\bar{\omega} (\rho_2 \sqrt{E} U - \rho_1 \sqrt{G} V)}{\rho_2 - \rho_1}.$$

In order that these functions satisfy equations (20), the function ω must be such that

$$(33) \quad \begin{aligned} \frac{\partial \log \bar{\omega}}{\partial u} + \frac{\sqrt{\bar{E}}}{\sqrt{\bar{G}}} \frac{U}{V} \frac{\partial}{\partial u} \left(\frac{1}{\rho_1} \right) \rho_1 \rho_2 - \frac{\partial}{\partial u} \log \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) &= 0, \\ \frac{\partial \log \bar{\omega}}{\partial v} - \frac{\sqrt{\bar{G}}}{\sqrt{\bar{E}}} \frac{V}{U} \frac{\partial}{\partial v} \left(\frac{1}{\rho_2} \right) \rho_1 \rho_2 - \frac{\partial}{\partial v} \log \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) &= 0. \end{aligned}$$

Expressing the consistency of these equations, we obtain

$$(34) \quad \frac{\partial}{\partial v} \left(\frac{\sqrt{\bar{E}}}{\sqrt{\bar{G}}} \frac{U}{V} \frac{\partial}{\partial u} \left(\frac{1}{\rho_1} \right) \rho_1 \rho_2 \right) + \frac{\partial}{\partial u} \left(\frac{\sqrt{\bar{G}}}{\sqrt{\bar{E}}} \frac{V}{U} \frac{\partial}{\partial v} \left(\frac{1}{\rho_2} \right) \rho_1 \rho_2 \right) = 0,$$

which is the characteristic equation of surfaces Ω given by Demoulin* without proof.

3. PROPERTIES OF SURFACES C . DETERMINATION OF SURFACES C_0

From the preceding section we have

THEOREM I. *A surface C is characterized by the property that it possesses a conjugate system with equal point invariants and that the point equation admits a solution t such that $x^2 + y^2 + z^2 - t^2$ also is a solution, where x, y , and z are the cartesian coördinates of C .*

We shall investigate these surfaces C , and understand that in what follows the conjugate system 2, 0 with equal point invariants is parametric.

If we put

$$\begin{aligned} \left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial u} \right)^2 - \left(\frac{\partial t}{\partial u} \right)^2 &= \sigma^2, \\ \left(\frac{\partial x}{\partial v} \right)^2 + \left(\frac{\partial y}{\partial v} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2 - \left(\frac{\partial t}{\partial v} \right)^2 &= \tau^2, \end{aligned}$$

it follows that

$$\frac{\partial}{\partial v} \log \sigma \sqrt{\rho} = 0, \quad \frac{\partial}{\partial u} \log \tau \sqrt{\rho} = 0.$$

Hence the parameters can be chosen so that

$$\sigma = \tau = 1/\sqrt{\rho}.$$

Accordingly we have

$$(35) \quad E - \left(\frac{\partial t}{\partial u} \right)^2 = \frac{1}{\rho}, \quad G - \left(\frac{\partial t}{\partial v} \right)^2 = \frac{1}{\rho}.$$

* Loc. cit., p. 707.

Since by hypothesis $x^2 + y^2 + z^2 - t^2$ is a solution of equation (1), we have also

$$(36) \quad F - \frac{\partial t}{\partial u} \frac{\partial t}{\partial v} = 0.$$

Equations (35) may be replaced by

$$(37) \quad \frac{\partial t}{\partial u} = \sqrt{E - 1/\rho}, \quad \frac{\partial t}{\partial v} = \sqrt{G - 1/\rho},$$

if the signs of the parameters u and v are suitably chosen.

Now equation (36) may be written

$$(38) \quad F = (E - 1/\rho)^{\frac{1}{2}} (G - 1/\rho)^{\frac{1}{2}}.$$

Moreover, the consistency of equations (37) requires that

$$(39) \quad \frac{\partial}{\partial v} (E - 1/\rho)^{\frac{1}{2}} = \frac{\partial}{\partial u} (G - 1/\rho)^{\frac{1}{2}}.$$

Equations (38) and (39) characterize a surface C . For, if they are satisfied, the function t defined by (37) satisfies (1) and $x^2 + y^2 + z^2 - t^2$ also is a solution in consequence of (38).

We have shown* that when the point equation of a surface is of the form (1) the following identities hold

$$(40) \quad \begin{aligned} \frac{\partial \sqrt{E}}{\partial v} &= -\sqrt{E} \frac{\partial \log \sqrt{\rho}}{\partial v} - \sqrt{G} \cos 2\omega \frac{\partial \log \sqrt{\rho}}{\partial u}, \\ \frac{\partial \sqrt{G}}{\partial u} &= -\sqrt{G} \frac{\partial \log \sqrt{\rho}}{\partial u} - \sqrt{E} \cos 2\omega \frac{\partial \log \sqrt{\rho}}{\partial v}, \end{aligned}$$

where 2ω is the angle between the parametric curves. Also in this case the Christoffel symbols formed with respect to the linear element of the surface represent the following expressions:

$$(41) \quad \begin{aligned} \{11\} &= \frac{\partial \log \sqrt{E}}{\partial u} + 2 \cot 2\omega \frac{\partial \omega}{\partial u} - \frac{\sqrt{E}}{\sqrt{G}} \cos 2\omega \frac{\partial \log \sqrt{\rho}}{\partial v}, \\ \{22\} &= \frac{E}{G} \frac{\partial \log \sqrt{\rho}}{\partial v} - 2 \frac{\sqrt{E}}{\sqrt{G}} \frac{1}{\sin 2\omega} \frac{\partial \omega}{\partial u}, \\ \{12\} &= \frac{G}{E} \frac{\partial \log \sqrt{\rho}}{\partial u} - 2 \frac{\sqrt{G}}{\sqrt{E}} \frac{1}{\sin 2\omega} \frac{\partial \omega}{\partial v}, \\ \{21\} &= \frac{\partial \log \sqrt{G}}{\partial v} + 2 \cot 2\omega \frac{\partial \omega}{\partial v} - \frac{\sqrt{G}}{\sqrt{E}} \cos 2\omega \frac{\partial \log \sqrt{\rho}}{\partial u}. \end{aligned}$$

* M., p. 419.

We suppose that a surface C is known and we seek a conjugate surface C_0 in the sense of § 2. To this end we introduce three functions a_0 , b_0 , T_0 by the equations

$$(42) \quad \begin{aligned} a_0 &= -\frac{T_0}{2 \cos \omega} \left(\frac{1}{\sqrt{E}} \frac{\partial t}{\partial u} + \frac{1}{\sqrt{G}} \frac{\partial t}{\partial v} \right), \\ b_0 &= \frac{T_0}{2 \sin \omega} \left(\frac{1}{\sqrt{E}} \frac{\partial t}{\partial u} - \frac{1}{\sqrt{G}} \frac{\partial t}{\partial v} \right). \end{aligned}$$

If in the first four of equations (10) we replace the subscript 1 by 0 and require that a_0 and b_0 as given by (42) shall satisfy these equations, the resulting equations of condition are reducible by means of (40) and (41) to

$$(43) \quad \begin{aligned} \frac{\partial T_0}{\partial u} &= m_0 (\lambda_0 - \rho \theta_0) \frac{\partial t}{\partial u}, & \frac{\partial T_0}{\partial v} &= m_0 (\lambda_0 + \rho \theta_0) \frac{\partial t}{\partial v}, \\ w_0 D &= \frac{m_0}{\rho} (\lambda_0 - \rho \theta_0) + T_0 \left(\{11\} \frac{\partial t}{\partial u} + \{21\} \frac{\partial t}{\partial v} - \frac{\partial^2 t}{\partial u^2} \right), \\ w_0 D'' &= \frac{m_0}{\rho} (\lambda_0 + \rho \theta_0) + T_0 \left(\{12\} \frac{\partial t}{\partial u} + \{22\} \frac{\partial t}{\partial v} - \frac{\partial^2 t}{\partial v^2} \right). \end{aligned}$$

From the definition of the Christoffel symbols* it follows that

$$E\{11\} + F\{12\} = \frac{1}{2} \frac{\partial E}{\partial u}, \quad F\{12\} + G\{22\} = \frac{1}{2} \frac{\partial G}{\partial v},$$

$$\frac{\partial}{\partial u} \log H \sqrt{\rho} = \{11\}, \quad \frac{\partial}{\partial v} \log H \sqrt{\rho} = \{22\},$$

where

$$H = \sqrt{EG - F^2} = \sqrt{EG} \sin 2\omega.$$

In consequence of these identities and equations (35) and (36) we have

$$(44) \quad \begin{aligned} \{11\} \frac{\partial t}{\partial u} + \{21\} \frac{\partial t}{\partial v} - \frac{\partial^2 t}{\partial u^2} &= - \left(\{11\} + \frac{\partial \log \sqrt{\rho}}{\partial u} \right) / \rho \frac{\partial t}{\partial u} \\ &= - \frac{\partial}{\partial u} \log H \rho / \rho \frac{\partial t}{\partial u}, \\ \{12\} \frac{\partial t}{\partial u} + \{22\} \frac{\partial t}{\partial v} - \frac{\partial^2 t}{\partial v^2} &= - \left(\{22\} + \frac{\partial \log \sqrt{\rho}}{\partial v} \right) / \rho \frac{\partial t}{\partial v} \\ &= - \frac{\partial}{\partial v} \log H \rho / \rho \frac{\partial t}{\partial v}, \end{aligned}$$

and

$$(45) \quad G \frac{\partial t}{\partial u} - F \frac{\partial t}{\partial v} = \frac{1}{\rho} \frac{\partial t}{\partial u}, \quad F \frac{\partial t}{\partial u} - E \frac{\partial t}{\partial v} = -\frac{1}{\rho} \frac{\partial t}{\partial v}.$$

* E., pp. 152, 153.

If in the last two of equations (10) we replace the subscript 1 by 0 and substitute for w_0 the expression

$$(46) \quad w_0 = T_0/H\rho,$$

the resulting equations are reducible by means of (44) and (45) to

$$(47) \quad \begin{aligned} \frac{DT_0}{H} &= m_0(\lambda_0 - \rho\theta_0) - T_0 \left/ \frac{\partial t}{\partial u} \cdot \frac{\partial}{\partial u} \log H\rho, \right. \\ \frac{D''T_0}{H} &= m_0(\lambda_0 + \rho\theta_0) - T_0 \left/ \frac{\partial t}{\partial v} \cdot \frac{\partial}{\partial v} \log H\rho, \right. \end{aligned}$$

which in consequence of (44) are equivalent to the last two of equations (43).

If we substitute in the first two of equations (43) the values of $\lambda_0 - \rho\theta_0$ and $\lambda_0 + \rho\theta_0$ given by (47), the resulting equations are reducible by means of (46) to

$$(48) \quad \frac{\partial}{\partial u} \log w_0 = \frac{D}{H} \frac{\partial t}{\partial u}, \quad \frac{\partial}{\partial v} \log w_0 = \frac{D''}{H} \frac{\partial t}{\partial v}.$$

With the aid of the Codazzi equations* for C the condition of integrability of equations (48) is reducible to

$$(49) \quad \begin{aligned} \frac{D}{H} \left[\left(\frac{\partial \log \sqrt{\rho}}{\partial v} + \left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\} \right) \frac{\partial t}{\partial u} + \left(\frac{\partial \log \sqrt{\rho}}{\partial u} - \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\} \right) \frac{\partial t}{\partial v} \right] \\ = \frac{D''}{H} \left[\left(\frac{\partial \log \sqrt{\rho}}{\partial u} + \left\{ \begin{smallmatrix} 11 \\ 1 \end{smallmatrix} \right\} \right) \frac{\partial t}{\partial v} + \left(\frac{\partial \log \sqrt{\rho}}{\partial v} - \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\} \right) \frac{\partial t}{\partial u} \right]. \end{aligned}$$

Because of (35), (36), and (44) the expression in the first parenthesis can be given the form

$$- \rho \frac{\partial t}{\partial v} \left(E \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\} + F \left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\} - \frac{\partial t}{\partial u} \frac{\partial^2 t}{\partial v^2} - \frac{1}{\rho} \frac{\partial \log \sqrt{\rho}}{\partial u} \right).$$

From the general definition of the Christoffel symbols it follows that

$$E \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\} + F \left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\} = \frac{\partial F}{\partial v} - \frac{1}{2} \frac{\partial G}{\partial u}.$$

Hence for the special values (35) and (36) the quantity in the parenthesis vanishes identically. The same is true of the coefficient of D''/H in (49). Recapitulating we see that the function w_0 is given by the quadratures (48), and the other functions a_0 , b_0 , λ_0 , θ_0 follow directly from (42), (46), and (47). Hence we have

THEOREM II. *When a surface C is known, a set of solutions of equations (1), (2), and (10) can be found by a quadrature; the corresponding transform of C*

* Cf. E., p. 156.

is a surface C_0 such that lines joining corresponding points on C and C_0 form a normal congruence.*

4. FUNDAMENTAL FUNCTIONS OF SURFACES Ω

In this section we determine the expressions for the fundamental coefficients of a surface Ω in terms of the functions for C and the transformation functions by which C_0 is obtained.

If \bar{S} denotes a surface normal to the congruence of joins of corresponding points on C and C_0 , its coördinates, \bar{x} , \bar{y} , \bar{z} , are expressible thus:

$$(50) \quad \bar{x} = x + t\bar{X}, \quad \bar{y} = y + t\bar{Y}, \quad \bar{z} = z + t\bar{Z},$$

where as follows from (14) and (9)

$$(51) \quad \bar{X} = \frac{x_0 - x}{T_0} \lambda_0 m_0 = \frac{1}{T_0} (a_0 X_1 + b_0 X_2 + w_0 X),$$

and similar expressions for \bar{Y} and \bar{Z} .

With the aid of (3) and (43) we obtain from the first of these expressions for \bar{X} the following:

$$(52) \quad \begin{aligned} \frac{\partial \bar{X}}{\partial u} &= -\frac{m_0}{T_0} (\lambda_0 - \rho\theta_0) \left(\frac{\partial x}{\partial u} + \frac{\partial t}{\partial u} \bar{X} \right), \\ \frac{\partial \bar{X}}{\partial v} &= -\frac{m_0}{T_0} (\lambda_0 + \rho\theta_0) \left(\frac{\partial x}{\partial v} + \frac{\partial t}{\partial v} \bar{X} \right). \end{aligned}$$

Making use of these results, we derive from (50)

$$(53) \quad \begin{aligned} \frac{\partial \bar{x}}{\partial u} &= \left[1 - \frac{m_0 t}{T_0} (\lambda_0 - \rho\theta_0) \right] \left(\frac{\partial x}{\partial u} + \frac{\partial t}{\partial u} \bar{X} \right), \\ \frac{\partial \bar{x}}{\partial v} &= \left[1 - \frac{m_0 t}{T_0} (\lambda_0 + \rho\theta_0) \right] \left(\frac{\partial x}{\partial v} + \frac{\partial t}{\partial v} \bar{X} \right). \end{aligned}$$

A comparison of (52) and (53) shows that the principal radii of normal curvature of \bar{S} are given by

$$\rho_1 = \frac{T_0}{m_0(\lambda_0 - \rho\theta_0)} - t, \quad \rho_2 = \frac{T_0}{m_0(\lambda_0 + \rho\theta_0)} - t.$$

From (35), (36), (42), and (51) we obtain the following identities

$$\begin{aligned} \Sigma \left(\frac{\partial x}{\partial u} + \frac{\partial t}{\partial u} \bar{X} \right)^2 &= \Sigma \left(\frac{\partial x}{\partial v} + \frac{\partial t}{\partial v} \bar{X} \right)^2 = \frac{1}{\rho}, \\ \Sigma \left(\frac{\partial x}{\partial u} + \frac{\partial t}{\partial u} \bar{X} \right) \left(\frac{\partial x}{\partial v} + \frac{\partial t}{\partial v} \bar{X} \right) &= 0. \end{aligned}$$

* The limiting case when C_0 is at infinity is considered in § 12.

Hence if \bar{E} , \bar{F} , \bar{G} and \bar{D} , \bar{D}' , \bar{D}'' denote the first and second fundamental coefficients of \bar{S} , we have from (52) and (53)

$$\begin{aligned}
 \sqrt{\bar{E}} &= \left[1 - \frac{m_0 t}{T_0} (\lambda_0 - \rho \theta_0) \right] \frac{1}{\sqrt{\rho}}, \\
 \sqrt{\bar{G}} &= \left[1 - \frac{m_0 t}{T_0} (\lambda_0 + \rho \theta_0) \right] \frac{1}{\sqrt{\rho}}, \\
 \bar{D} &= \frac{m_0}{T_0} (\lambda_0 - \rho \theta_0) \frac{\sqrt{\bar{E}}}{\sqrt{\rho}}, \quad D'' = \frac{m_0}{T_0} (\lambda_0 + \rho \theta_0) \frac{\sqrt{\bar{G}}}{\sqrt{\rho}}, \\
 \bar{F} &= \bar{D}' = 0.
 \end{aligned}
 \tag{54}$$

If we make use of equations analogous to (2) and (12), we can obtain without difficulty from the above values for ρ_1 and ρ_2 the expressions

$$\begin{aligned}
 \frac{\partial \rho_1}{\partial u} &= \frac{T_0}{m_0 (\lambda_0 - \rho \theta_0)^2} \left(2\rho \frac{\partial \theta_0}{\partial u} + \theta_0 \frac{\partial \rho}{\partial u} \right), \\
 \frac{\partial \rho_2}{\partial v} &= - \frac{T_0}{m_0 (\lambda_0 + \rho \theta_0)^2} \left(2\rho \frac{\partial \theta_0}{\partial v} + \theta_0 \frac{\partial \rho}{\partial v} \right).
 \end{aligned}$$

It is readily shown that equation (34) is satisfied by these expressions.

5. DETERMINATION OF THE SURFACES C'_0

In § 2 we remarked that \bar{S} is one of the sheets of the envelope of spheres with centers on C and radii equal to t . Let \bar{S}' denote the other sheet. Evidently it also is a surface Ω , on account of the symmetry of the problem. If so, there is a surface C'_0 conjugate to C , such that the lines joining corresponding points on C and C'_0 are the normals to \bar{S}' (cf. § 2). We seek now the values of the functions a'_0 , b'_0 , w'_0 , θ'_0 , λ'_0 , which determine this special transformation.

From equations (15) it follows that

$$a'_0/T'_0 = a_0/T_0, \quad b'_0/T'_0 = b_0/T_0.$$

From the definition of T_0 and T'_0 (cf. equation (12)) it is necessary that

$$w'_0/w_0 = \pm T'_0/T_0.$$

If we take the upper sign we have

$$a'_0/a_0 = b'_0/b_0 = w'_0/w_0 = T'_0/T_0 = \sigma,$$

where σ denotes a factor of proportionality, which is found to be a constant when these values of a'_0 , b'_0 , w'_0 are substituted in

$$(55) \quad \begin{aligned} \frac{\partial w'_0}{\partial u} &= -\frac{D}{2\sqrt{E}} \left(\frac{a'_0}{\cos \omega} - \frac{b'_0}{\sin \omega} \right), \\ \frac{\partial w'_0}{\partial v} &= -\frac{D''}{2\sqrt{G}} \left(\frac{a'_0}{\cos \omega} + \frac{b'_0}{\sin \omega} \right), \end{aligned}$$

equations analogous to the last two of (10). From (43) it follows that

$$\lambda'_0 = \sigma \lambda_0, \quad \theta'_0 = \sigma \theta_0, \quad m'_0 = m_0,$$

and from equations analogous to (9) we see that in this case C_0 and C'_0 coincide.

We consider now the other possibility of sign, and have

$$a'_0/a_0 = b'_0/b_0 = -w'_0/w_0 = T'_0/T_0 = \sigma.$$

When these values are substituted in (55), the resulting equations are reducible to

$$\frac{\partial}{\partial u} \log \sigma w_0^2 = 0, \quad \frac{\partial}{\partial v} \log \sigma w_0^2 = 0.$$

Hence σw_0^2 is a constant, which may be taken equal to unity in all generality and so we have

$$(56) \quad a'_0 = a_0/w_0^2, \quad b'_0 = b_0/w_0^2, \quad w'_0 = -1/w_0, \quad T'_0 = T_0/w_0^2.$$

From the last two of these equations and (46) we get

$$(57) \quad w'_0 = -T'_0/H\rho,$$

which is consistent with equations analogous to (48) and those from which it was derived, as there indicated. It is evident that the values for a'_0 and b'_0 satisfy equations analogous to (42).

When the expression (57) for T'_0 is substituted in

$$\frac{\partial T'_0}{\partial u} = m'_0(\lambda'_0 - \rho\theta'_0) \frac{\partial t}{\partial u}, \quad \frac{\partial T'_0}{\partial v} = m'_0(\lambda'_0 + \rho\theta'_0) \frac{\partial t}{\partial v},$$

the resulting equations are reducible by (43), (48), and (56) to

$$(58) \quad \begin{aligned} m'_0(\lambda'_0 - \rho\theta'_0) &= \frac{m_0}{w_0^2}(\lambda_0 - \rho\theta_0) - \frac{2T_0}{w_0^2} \frac{D}{H}, \\ m'_0(\lambda'_0 + \rho\theta'_0) &= \frac{m_0}{w_0^2}(\lambda_0 + \rho\theta_0) - \frac{2T_0}{w_0^2} \frac{D''}{H}. \end{aligned}$$

From these with the aid of (47) we obtain

$$(9) \quad \begin{aligned} m'_0(\lambda'_0 - \rho\theta'_0) &= -\frac{m_0}{w_0^2}(\lambda_0 - \rho\theta_0) + 2T_0 \Big/ w_0^2 \frac{\partial t}{\partial u} \cdot \frac{\partial}{\partial u} \log H\rho, \\ m'_0(\lambda'_0 + \rho\theta'_0) &= -\frac{m_0}{w_0^2}(\lambda_0 + \rho\theta_0) + 2T_0 \Big/ w_0^2 \frac{\partial t}{\partial v} \cdot \frac{\partial}{\partial v} \log H\rho. \end{aligned}$$

It is readily shown that these values satisfy equations for C'_0 analogous to the last two of (43).

By making use of (46) we obtain from (58)

$$\begin{aligned} m'_0 \lambda'_0 &= \frac{m_0}{w_0^2} \lambda_0 - \frac{\rho}{w_0} (D'' + D), \\ m'_0 \theta'_0 &= \frac{m_0}{w_0^2} \theta_0 - \frac{1}{w_0} (D'' - D). \end{aligned} \quad (60)$$

With the aid of equations (48) and the Codazzi equations* for C we show that these values for λ'_0 and θ'_0 satisfy the equations

$$\frac{\partial \lambda'_0}{\partial u} = -\rho \frac{\partial \theta'_0}{\partial u}, \quad \frac{\partial \lambda'_0}{\partial v} = \rho \frac{\partial \theta'_0}{\partial v}.$$

Hence the transformation functions as given by (56), (57), and (60) satisfy all the necessary and sufficient conditions, and we have

THEOREM III. *With a surface C there are associated two unique surfaces C_0 and C'_0 such that the lines joining points on either of these surfaces and the corresponding points of C form a congruence normal to a surface Ω .*

The determination of these conjugate surfaces can be effected without quadrature in consequence of the statement at the beginning of the present section, but the determination of the functions λ , θ , a , b , w requires a quadrature.

6. TRANSFORMATIONS K_m OF SURFACES C

We establish now transformations K of a surface C into surfaces of the same kind. Let C_1 be one of the new surfaces and θ_1 the solution of equation (1) determining this transformation. From § 1 it follows that the coördinates x_1 , y_1 , z_1 are given by equations of the form (3). It is evident that t_1 , determined by the equations

$$\begin{aligned} \frac{\partial t_1}{\partial u} &= \frac{\rho}{\lambda_1} \left[(t_1 - t) \frac{\partial \theta_1}{\partial u} + \theta_1 \frac{\partial t}{\partial u} \right], \\ \frac{\partial t_1}{\partial v} &= -\frac{\rho}{\lambda_1} \left[(t_1 - t) \frac{\partial \theta_1}{\partial v} + \theta_1 \frac{\partial t}{\partial v} \right], \end{aligned} \quad (61)$$

is a solution of the point equation of C_1 , namely (4). By the definition of the surface C the function $x^2 + y^2 + z^2 - t^2$ also is a solution of equation (1). If the equations which result when t and t_1 in (61) are replaced by $x^2 + y^2 + z^2 - t^2$ and $x_1^2 + y_1^2 + z_1^2 - t_1^2$, respectively, are satisfied, then C_1 is a surface of the same kind as C . When we express this condition, the resulting equations are reducible to

* E., p. 155.

$$(62) \quad \begin{aligned} W_1 \frac{\partial \theta_1}{\partial u} + 2\theta_1 \left(\sum (x_1 - x) \frac{\partial x}{\partial u} - (t_1 - t) \frac{\partial t}{\partial u} \right) &= 0, \\ W_1 \frac{\partial \theta_1}{\partial v} + 2\theta_1 \left(\sum (x_1 - x) \frac{\partial x}{\partial v} - (t_1 - t) \frac{\partial t}{\partial v} \right) &= 0, \end{aligned}$$

where we have put

$$(63) \quad W_1 = \sum (x_1 - x)^2 - (t_1 - t)^2 = (T_1/\lambda_1 m_1)^2 - (t_1 - t)^2,$$

the latter being a consequence of (9) and (12) and \sum indicating as usual summation with respect to x, y , and z .

By means of (61) and the formulas of § 1 we find

$$\begin{aligned} \frac{\partial W_1}{\partial u} &= \frac{\lambda_1 + \rho\theta_1}{\lambda_1 \theta_1} \frac{\partial \theta_1}{\partial u} W_1 = \frac{1}{\lambda_1 \theta_1} \left(\lambda_1 \frac{\partial \theta_1}{\partial u} - \theta_1 \frac{\partial \lambda_1}{\partial u} \right) W_1, \\ \frac{\partial W_1}{\partial v} &= \frac{\lambda_1 - \rho\theta_1}{\lambda_1 \theta_1} \frac{\partial \theta_1}{\partial v} W_1 = \frac{1}{\lambda_1 \theta_1} \left(\lambda_1 \frac{\partial \theta_1}{\partial v} - \theta_1 \frac{\partial \lambda_1}{\partial v} \right) W_1. \end{aligned}$$

Hence, to within a constant factor, W_1 is equal to θ_1/λ_1 . We consider later the case where this factor is equal to zero. For the case where the constant is not equal to zero we put

$$(64) \quad W_1 = 2\theta_1/\lambda_1 m_1.$$

In consequence of this result and of (7), (8), and (15) equations (62) may be put in the form

$$(65) \quad \begin{aligned} \frac{\partial \theta_1}{\partial u} + \sqrt{E} \left[(\cos \omega a_1 - \sin \omega b_1) + (t_1 - t) \frac{m_1 \lambda_1}{T_0} (\cos \omega a_0 \right. \\ \left. - \sin \omega b_0) \right] &= 0, \\ \frac{\partial \theta_1}{\partial v} + \sqrt{G} \left[(\cos \omega a_1 + \sin \omega b_1) + (t_1 - t) \frac{m_1 \lambda_1}{T_0} (\cos \omega a_0 \right. \\ \left. + \sin \omega b_0) \right] &= 0. \end{aligned}$$

Thus it is seen that by the above choice of the constant of integration in (64) the constant m_1 does not now appear with both θ_1 and λ_1 , as it does in (9), and consequently it is a significant constant. It is readily found that θ_1 , as given by (65), satisfies equation (1), in consequence of (9), (61), and (43).

From (63), (64), and (12) we have

$$(65') \quad 2m_1 \theta_1 \lambda_1 = a_1^2 + b_1^2 + w_1^2 - m_1^2 \lambda_1^2 (t_1 - t)^2.$$

Hence we have

THEOREM IV. *When a surface C is known, each set of functions $\theta_1, \lambda_1, a_1, b_1, w_1, t_1$ satisfying the completely integrable system of equations (2), (10),*

(61), and (65), and in the relation (65'), determines a transformation of C into a surface of the same kind.

The general solution of this system of equations involves five essential constants in addition to m_1 .

The transformation from C into C_0 (cf. § 2) likewise satisfies these conditions. In fact, in this case equations (62) become

$$\begin{aligned} \left(\frac{T_0}{\lambda_0 m_0} + t_0 - t \right) & \left[\left(\frac{T_0}{\lambda_0 m_0} - t_0 + t \right) \frac{\partial \theta_0}{\partial u} \right. \\ & \left. + \frac{2\theta_0 \sqrt{E}}{T_0} (\cos \omega a_0 - \sin \omega b_0) \right] = 0, \\ (66) \quad \left(\frac{T_0}{\lambda_0 m_0} + t_0 - t \right) & \left[\left(\frac{T_0}{\lambda_0 m_0} - t_0 + t \right) \frac{\partial \theta_0}{\partial v} \right. \\ & \left. + \frac{2\theta_0 \sqrt{G}}{T_0} (\cos \omega a_0 + \sin \omega b_0) \right] = 0. \end{aligned}$$

This equation is satisfied if we have

$$(67) \quad t_0 = t - T_0 / \lambda_0 m_0,$$

as are also equations analogous to (61).

Since the surfaces C and C_0 bear similar relations to \bar{S} , we have analogous to (14) the equation

$$(68) \quad \bar{x} = x_0 + \frac{t_0 \lambda_0 m_0}{T_0} (x_0 - x),$$

which is consistent with (14) in consequence of (67).

Conversely we shall show that if the function t_0 of a surface C_0 arising from C by a transformation K_m is in the relation (67) to t for C , then the lines joining corresponding points on C and C_0 form a normal congruence. In fact, if we substitute this value of t_0 in (61) and make use of (13), we get equations (15), which are the necessary and sufficient condition that \bar{S} defined by (14) is normal to the joins of corresponding points on C and C_0 .

Comparing (63) and (67), we see that W is zero in this case. Moreover, from the results of the preceding paragraph, it follows that W is zero only in the case of the transformations giving the conjugate surfaces C_0 and C'_0 .

7. GENERALIZED THEOREM OF PERMUTABILITY OF GENERAL TRANSFORMATIONS K

In our previous paper* we established a theorem of permutability for transformations K . We shall recall the results and make a generalization

* M., p. 406.

which will be of service in the development of the theory of the transformations K for the kind of surfaces under discussion in the present paper. In this section we use the term surface S to denote any surface referred to a conjugate system with equal point invariants.

Suppose that θ_1 and θ_2 are two linearly independent solutions of (1) and that S_1 and S_2 are the surfaces arising from the surface S by the transformations K determined by these functions. The cartesian coördinates $x_1, y_1, z_1; x_2, y_2, z_2$ of S_1 and S_2 are given by equations of the form (3), namely

$$(69) \quad \begin{aligned} \frac{\partial x_i}{\partial u} &= \frac{\rho}{\lambda_i} \left[(x_i - x) \frac{\partial \theta_i}{\partial u} + \theta_i \frac{\partial x}{\partial u} \right] \\ \frac{\partial x_i}{\partial v} &= -\frac{\rho}{\lambda_i} \left[(x_i - x) \frac{\partial \theta_i}{\partial v} + \theta_i \frac{\partial x}{\partial v} \right] \end{aligned} \quad (i = 1, 2),$$

where in accordance with (2)

$$(70) \quad \frac{\partial \lambda_i}{\partial u} = -\rho \frac{\partial \theta_i}{\partial u}, \quad \frac{\partial \lambda_i}{\partial v} = \rho \frac{\partial \theta_i}{\partial v}.$$

From (4) it follows that the functions defined by (69) satisfy the corresponding equation

$$(71) \quad \frac{\partial^2 \phi_i}{\partial u \partial v} + \frac{\partial}{\partial v} \log \frac{\lambda_i}{\sqrt{\rho} \theta_i} \frac{\partial \phi_i}{\partial u} + \frac{\partial}{\partial u} \log \frac{\lambda_i}{\sqrt{\rho} \theta_i} \frac{\partial \phi_i}{\partial v} = 0 \quad (i = 1, 2).$$

These equations, for $i = 1$ and 2 , are satisfied also by the functions θ_{12} and θ_{21} respectively, defined by

$$(72) \quad \begin{aligned} \frac{\partial}{\partial u} (\theta_{ij} \lambda_i) &= \rho \left(\theta_i \frac{\partial \theta_j}{\partial u} - \theta_j \frac{\partial \theta_i}{\partial u} \right) \\ \frac{\partial}{\partial v} (\theta_{ij} \lambda_i) &= -\rho \left(\theta_i \frac{\partial \theta_j}{\partial v} - \theta_j \frac{\partial \theta_i}{\partial v} \right) \end{aligned} \quad (i, j = 1, 2; i \neq j).$$

As thus defined the functions θ_{12} and θ_{21} are determined only to within the additive functions c_i/λ_i respectively, where c_i is an arbitrary constant. Hereafter in speaking of two such functions we assume that the constants are so chosen that

$$(73) \quad \theta_{ij} \lambda_i + \theta_{ji} \lambda_j = 0 \quad (i \neq j),$$

which evidently is consistent with (72).

If we put

$$(74) \quad \rho_i = \lambda_i^2 / \rho \theta_i^2, \quad (i = 1, 2),$$

equations (71) are of the same form as equation (1). The functions λ_{12} and λ_{21} which together with θ_{12} and θ_{21} give transformations of S_1 and S_2 respectively are given by

$$(75) \quad \frac{\partial \lambda_{ij}}{\partial u} = -\frac{\lambda_i^2}{\rho \theta_i^2} \frac{\partial \theta_{ij}}{\partial u}, \quad \frac{\partial \lambda_{ij}}{\partial v} = \frac{\lambda_i^2}{\rho \theta_i^2} \frac{\partial \theta_{ij}}{\partial v} \quad (i, j = 1, 2; i \neq j).$$

We have shown* that these equations are satisfied by the functions λ_{ij} given by

$$(76) \quad \lambda_{ij} \theta_i = \theta_{ij} \lambda_i - \theta_j \lambda_i + \theta_i \lambda_j \quad (i \neq j).$$

Moreover, in consequence of (73) we have

$$(77) \quad \lambda_{ij} \theta_i + \lambda_{ji} \theta_j = 0.$$

When S_1 and S_2 are transformed by means of the functions θ_{12} , λ_{12} and θ_{21} , λ_{21} respectively, the resulting surfaces coincide.† In fact, if this surface be denoted by S_{12} , its cartesian coördinates x_{12} , y_{12} , z_{12} are given by equations of the form

$$(78) \quad \lambda_{12} \theta_1 x_{12} = \theta_{12} \lambda_1 x - \lambda_1 \theta_2 x_1 + \lambda_2 \theta_1 x,$$

or by the equivalent equations

$$(79) \quad \lambda_{21} \theta_2 x_{12} = \theta_{21} \lambda_2 x - \lambda_2 \theta_1 x_2 + \lambda_1 \theta_2 x_1.$$

We say that the surfaces S , S_1 , S_2 , S_{12} form a *quatern* in accordance with the theorem of permutability. From (78) it is evident that four corresponding points M , M_1 , M_2 , M_{12} on these respective surfaces are coplanar.

We extend these results by considering three linearly independent solutions of (1), say θ_1 , θ_2 , θ_3 , and we denote by S_1 , S_2 , and S_3 the corresponding transforms of S . Furthermore, we denote by S_{12} , S_{13} , and S_{23} the surfaces which form quaterns with the respective groups of surfaces, S , S_1 , S_2 ; S , S_1 , S_3 ; S , S_2 , S_3 . Since S_{12} and S_{13} are transforms of S_1 , we seek the surface S' which forms a quatern with them.

If we denote by θ'_{12} , λ'_{12} and θ'_{13} , λ'_{13} the functions by means of which S_{12} and S_{13} are transformed into S' , in accordance with (76) we must have

$$(80) \quad \lambda'_{12} \theta_{12} = \theta'_{12} \lambda_{12} - \lambda_{12} \theta_{13} + \lambda_{13} \theta_{12},$$

and analogously to (78),

$$(81) \quad \lambda'_{12} \theta_{12} x' = \theta'_{12} \lambda_{12} x_1 - \lambda_{12} \theta_{13} x_{12} + \lambda_{13} \theta_{12} x_{13}.$$

In consequence of (78) and a similar expression for x_{13} equation (81) is equivalent to

$$(82) \quad \lambda'_{12} \theta_{12} x' = (\theta'_{12} \lambda_{12} \theta_1 + \theta_{13} \lambda_1 \theta_2 - \theta_{12} \lambda_1 \theta_3) \frac{x_1}{\theta_1} - \theta_{13} \lambda_2 x_2 + \theta_{12} \lambda_3 x_3.$$

On the assumption that S' and each of the triples of surfaces S_2 , S_{21} , S_{23}

* M., p. 405.

† M., p. 406.

and S_3 , S_{31} , S_{32}^* form quaterns, the following equations analogous to (82) must hold

$$\begin{aligned} \lambda'_{21} \theta_{21} x' &= (\theta'_{21} \lambda_{21} \theta_2 + \theta_{23} \lambda_2 \theta_1 - \theta_{21} \lambda_2 \theta_3) \frac{x_2}{\theta_2} - \lambda_1 \theta_{23} x_1 + \lambda_3 \theta_{21} x_3, \\ \lambda'_{31} \theta_{31} x' &= (\theta'_{31} \lambda_{31} \theta_3 + \theta_{32} \lambda_3 \theta_1 - \theta_{31} \lambda_3 \theta_2) \frac{x_3}{\theta_3} - \lambda_1 \theta_{32} x_1 + \lambda_2 \theta_{31} x_2. \end{aligned} \quad (83)$$

If equation (82) and the first of (83) be multiplied by λ_1 and λ_2 respectively and the results be added, it follows in consequence of (73) that the coefficient of x_3 vanishes and consequently we have an expression of the form

$$Ax' + Bx_1 + Cx_2 = 0.$$

Since this equation holds also in the y 's and z 's, the quantities A , B , and C must be zero. Consequently we must have

$$\begin{aligned} \lambda'_{21} &= \lambda'_{12}, \\ \theta'_{12} \lambda_{12} \theta_1 &= -\theta_{13} \lambda_1 \theta_2 + \theta_{12} \lambda_1 \theta_3 + \theta_{23} \lambda_2 \theta_1, \\ \theta'_{21} \lambda_{21} \theta_2 &= -\theta_{23} \lambda_2 \theta_1 + \theta_{21} \lambda_2 \theta_3 + \theta_{13} \lambda_1 \theta_2. \end{aligned} \quad (84)$$

From the definition of the functions θ'_{ij} and λ'_{ij} it follows that

$$\theta'_{ij} = \theta'_{ji}, \quad \lambda'_{ij} = \lambda'_{ji} \cdot \dagger$$

Hence the first of (84) is true and the second and third are equivalent to one another in consequence of (73) and (77). Furthermore, if we compare the second of (84) with (78), we see that θ'_{12} defined by this equation is a solution of the point equation of S_{12} and from the general theory it necessarily satisfies equations analogous to (72), namely

$$\begin{aligned} \frac{\partial}{\partial u} (\theta'_{ij} \lambda_{ij}) &= \frac{\lambda_i^2}{\rho \theta_i^2} \left(\theta_{ij} \frac{\partial \theta_{ik}}{\partial u} - \theta_{ik} \frac{\partial \theta_{ij}}{\partial v} \right) \\ \frac{\partial}{\partial v} (\theta'_{ij} \lambda_{ij}) &= -\frac{\lambda_i^2}{\rho \theta_i^2} \left(\theta_{ij} \frac{\partial \theta_{ik}}{\partial v} - \theta_{ik} \frac{\partial \theta_{ij}}{\partial u} \right) \end{aligned} \quad (i \neq j \neq k). \quad (85)$$

If we proceed in like manner with (82) and the second of (83), multiplying them respectively by λ_1 and λ_3 , we are brought to the equations

$$\begin{aligned} \lambda_3 \lambda'_{31} \theta_{31} - \lambda_1 \lambda'_{12} \theta_{12} &= 0, \\ \theta'_{31} \lambda_{31} \theta_3 &= -\theta_{32} \lambda_3 \theta_1 + \theta_{31} \lambda_3 \theta_2 + \theta_{12} \lambda_1 \theta_3. \end{aligned} \quad (86)$$

By means of equations (73) and equations analogous to (80) we find that the first of (86) is a consequence of the second and of (84). Furthermore, as

* From the definition of a surface S_{ij} it follows that S_{ji} is the same surface.

† Cf. the previous footnote.

the second of (86) is analogous to the second of (84), it follows that θ'_{31} so defined satisfies equations (85). Hence all the conditions are satisfied and not only have we shown that S' exists, but the functions determining S' are given without quadrature. Now equations (82) and (83) are reducible to

$$\begin{aligned} x'(\theta_{12}\lambda_3 + \theta_{23}\lambda_2 - \theta_{13}\lambda_2) &= \theta_{23}\lambda_2 x_1 - \theta_{13}\lambda_2 x_2 + \theta_{12}\lambda_3 x_3, \\ (87) \quad x'(\theta_{12}\lambda_1 - \theta_{32}\lambda_1 + \theta_{31}\lambda_2) &= -\theta_{32}\lambda_1 x_1 + \theta_{31}\lambda_2 x_2 + \theta_{12}\lambda_1 x_3, \\ x'(\theta_{13}\lambda_1 - \theta_{23}\lambda_1 + \theta_{21}\lambda_3) &= -\theta_{23}\lambda_1 x_1 + \theta_{13}\lambda_1 x_2 + \theta_{21}\lambda_3 x_3, \end{aligned}$$

which are equivalent to one another.

Hence we have the following generalized theorem of permutability:

THEOREM V. *If S is any surface referred to a conjugate system with equal point invariants, and S_1, S_2, S_3 are three surfaces of the same sort obtained from S by transformations K , and S_{12}, S_{13}, S_{23} are surfaces which together with the respective groups $S, S_1, S_2; S, S_1, S_3; S, S_2, S_3$ form quaterns, there exists a surface S' such that $(S_1, S_{12}, S_{13}, S'), (S_2, S_{12}, S_{23}, S'), (S_3, S_{13}, S_{23}, S')$ are quaterns. Moreover, when S_{12}, S_{13}, S_{23} are known, S' can be found without any quadratures.*

8. TRANSFORMATIONS OF SURFACES Ω

We consider now two conjugate surfaces C and C_0 . It is our purpose to show that, if C_1 is a transform of C in the sense of § 6, there exists a surface C_{10} , such that the surfaces C, C_1, C_0, C_{10} form a quatern (cf. § 7) and that C_{10} is conjugate to C_1 ; that is, the lines joining corresponding points on C_1 and C_{10} are normal to a family of surfaces Ω .

From (78) it follows that the coördinates x_{10}, y_{10}, z_{10} of C_{10} are given by equations of the form

$$(88) \quad \lambda_{01}\theta_0 x_{10} = \theta_{01}\lambda_0 x - \lambda_0\theta_1 x_0 + \lambda_1\theta_0 x_1,$$

where θ_{01} and λ_{01} are the functions by which C_0 is transformed into C_{10} . From this it is evident that the function t_{10} , defined by

$$(89) \quad \lambda_{01}\theta_0 t_{10} = \theta_{01}\lambda_0 t - \lambda_0\theta_1 t_0 + \lambda_1\theta_0 t_1,$$

is a solution of the point equation of C_{10} .

We know that the functions

$$\sum x^2 - t^2, \quad \sum x_1^2 - t_1^2, \quad \sum x_0^2 - t_0^2,$$

where the summation is with respect to the three coördinates, satisfy the point equations of C, C_1 , and C_0 respectively. We wish to show that

$$\lambda_{01}\theta_0(\sum x_{10}^2 - t_{10}^2) = \theta_{01}\lambda_0(\sum x^2 - t^2) - \lambda_0\theta_1(\sum x_0^2 - t_0^2) + \lambda_1\theta_0(\sum x_1^2 - t_1^2),$$

which is of the form of (88).

If we substitute in the left-hand member of this equation the expressions for x_{10} , y_{10} , z_{10} , t_{10} given by (88) and (89), and make use of the identity (cf. (76))

$$(90) \quad \lambda_{01} \theta_0 = \lambda_0 \theta_{01} - \lambda_0 \theta_1 + \lambda_1 \theta_0,$$

we get

$$(91) \quad \theta_{01} \{ \lambda_0 \theta_1 [\sum (x_0 - x)^2 - (t_0 - t)^2] - \lambda_1 \theta_0 [\sum (x_1 - x)^2 - (t_1 - t)^2] \} + \lambda_1 \theta_0 \theta_1 [\sum (x_1 - x_0)^2 - (t_1 - t_0)^2] = 0.$$

From equation (9) and an analogous expression for x_0 we obtain by subtraction

$$x_1 - x_0 = \left(\frac{a_1}{\lambda_1 m_1} - \frac{a_0}{\lambda_0 m_0} \right) X_1 + \left(\frac{b_1}{\lambda_1 m_1} - \frac{b_0}{\lambda_0 m_0} \right) X_2 + \left(\frac{w_1}{\lambda_1 m_1} - \frac{w_0}{\lambda_0 m_0} \right) X.$$

Hence with the aid of (67) we have

$$\begin{aligned} \sum (x_1 - x_0)^2 - (t_1 - t_0)^2 &= \frac{T_1^2}{\lambda_1^2 m_1^2} \\ &\quad - (t_1 - t)^2 - 2 \frac{T_0}{\lambda_0 m_0} (t_1 - t) - 2 \frac{\Phi_{10}}{\lambda_0 \lambda_1 m_0 m_1}, \end{aligned}$$

where we have put

$$(92) \quad \Phi_{10} = a_1 a_0 + b_1 b_0 + w_1 w_0.$$

From the expression for x_0 of the form (9) and from (67) it follows also that

$$\sum (x_0 - x)^2 - (t_0 - t)^2 = 0.$$

With the aid of these results and equations (63) and (64) we can reduce (91) to

$$(93) \quad m_0 \lambda_0 (\theta_{01} - \theta_1) + T_0 m_1 \lambda_1 (t_1 - t) + \Phi_{10} = 0.$$

It is necessary that θ_{01} given by (93) shall satisfy equations (72) with $i = 0$, $j = 1$. In order to establish this result we calculate first the derivatives of Φ_{10} . Making use of equations (10) and similar equations for a_0 , b_0 , w_0 , we find

$$\begin{aligned} \frac{\partial \Phi_{10}}{\partial u} &= \sqrt{E} \{ m_1 (a_0 \cos \omega - b_0 \sin \omega) (\rho \theta_1 - \lambda_1) \\ &\quad + m_0 (a_1 \cos \omega - b_1 \sin \omega) (\rho \theta_0 - \lambda_0) \}, \\ (94) \quad \frac{\partial \Phi_{10}}{\partial v} &= -\sqrt{G} \{ m_1 (a_0 \cos \omega + b_0 \sin \omega) (\rho \theta_1 + \lambda_1) \\ &\quad + m_0 (a_1 \cos \omega + b_1 \sin \omega) (\rho \theta_0 + \lambda_0) \}. \end{aligned}$$

With the aid of these equations and (64) it is readily found that θ_{01} satisfies the above mentioned conditions identically. Hence this value of θ_{01} serves to determine two surfaces C_1 and C_{10} which with C and C_0 form a quatern.

It remains to be shown that C_{10} is a conjugate surface of C_1 , which necessitates the condition

$$(95) \quad t_{10} = t_1 - T_{10}/\lambda_{10} m_0,$$

and

$$(96) \quad \frac{\partial T_{10}}{\partial u} = m_0 \left(\lambda_{10} - \frac{\lambda_1^2 \theta_{10}}{\rho \theta_1^2} \right) \frac{\partial t_1}{\partial u}, \quad \frac{\partial T_{10}}{\partial v} = m_0 \left(\lambda_{10} + \frac{\lambda_1^2 \theta_{10}}{\rho \theta_1^2} \right) \frac{\partial t_1}{\partial v},$$

from (67) and (43).

From (89), (67), and (95) we have, because of the identity (76) for $i = 0$, $j = 1$,

$$(97) \quad (T_{10} - T_0) \theta_1 + (\theta_{01} - \theta_1) m_0 \lambda_0 (t_1 - t) = 0.$$

With the aid of (90), (76), and (72), we find that T_{10} defined by (97) satisfies (96) identically. Hence all the conditions are satisfied and the lines joining corresponding points are normal to a surface Ω . Consequently we have

THEOREM VI. *If C and C_0 are two conjugate surfaces and C_1 is a surface obtained from C by a transformation K_{m_1} , a surface C_{10} can be found without quadratures which is conjugate to C_1 and in the relation of a transformation K_{m_1} with C_0 .*

In accordance with the general theory (§ 7) the lines MM_0 and $M_1 M_{10}$ lie in a plane. Let ξ , η , ζ denote the cartesian coördinates of the point P of intersection of these lines. Evidently

$$(98) \quad \xi = x + \sigma \frac{\lambda_0 m_0}{T_0} (x_0 - x) = x_1 + \frac{\sigma_1 \lambda_{10} m_0}{T_{10}} (x_{10} - x_1),$$

where σ and σ_1 denote the distances from M and M_1 respectively to P .

If x_{10} be replaced by its value from (88), the equality of the two expressions for ξ gives

$$x \left(1 - \frac{\sigma \lambda_0 m_0}{T_0} + \frac{\sigma_1 \lambda_0 m_0 \theta_{01}}{\theta_1 T_{10}} \right) + x_0 \lambda_0 m_0 \left(\frac{\sigma}{T_0} + \frac{\sigma_1}{T_{10}} \right) - x_1 \left(1 + \sigma_1 \lambda_0 m_0 \frac{-\theta_{01} + \theta_1}{\theta_1 T_{10}} \right) = 0.$$

Since similar equations hold in the y 's and z 's, the expressions in the parentheses must be zero. These equations are equivalent to

$$(99) \quad \sigma = - \frac{T_0 \theta_1}{m_0 \lambda_0 (\theta_{01} - \theta_1)}, \quad \sigma_1 = - \frac{T_{10} \theta_1}{m_0 \lambda_0 (\theta_{01} - \theta_1)}.$$

Equations (68) define a surface \bar{S} normal to the congruence determined by C and C_0 . A surface \bar{S}_1 normal to the congruence determined by C_1 and C_{10} is defined by similar equations. The distances from P to corresponding points \bar{M} and \bar{M}_1 on these surfaces are $t - \sigma$ and $t_1 - \sigma_1$ respectively.

In consequence of (97) and (99) we have

$$(100) \quad t - \sigma = t + \frac{T_0 \theta_1}{m_0 \lambda_0 (\theta_{01} - \theta_1)} = t_1 + \frac{T_{10} \theta_1}{m_0 \lambda_0 (\theta_{01} - \theta_1)} = t_1 - \sigma_1.$$

Consequently the surfaces \bar{S} and \bar{S}_1 are the envelope of a two-parameter family of spheres with lines of curvature in correspondence on the two sheets of the envelope. Since such envelopes of spheres were considered first by Ribaucour,* we say that \bar{S}_1 is obtained from \bar{S} by a transformation of the Ribaucour type. We have just seen that \bar{S}_1 is found by direct processes after S_1 has been determined. In consequence of Theorem IV we have

THEOREM VII. *Surfaces Ω admit transformations of the Ribaucour type into surfaces Ω , and the determination of these transformations requires the solution of the completely integrable set of equations (2), (10), (61), and (65).*

For the sake of brevity we say that \bar{S}_1 is obtained from \bar{S} by a transformation A_{m_1} .

When the values of σ and σ_1 , given by (99) are substituted in (98), the latter become

$$(101) \quad \xi = x + \frac{\theta_1}{\theta_{01} - \theta_1} (x - x_0) = \frac{\theta_{01} x - \theta_1 x_0}{\theta_{01} - \theta_1}.$$

In § 5 we remarked that the spheres with centers on C and radii determined by the corresponding values of t are enveloped by \bar{S} and a second surface \bar{S}' , which arises from \bar{S} by a transformation of the Ribaucour type. In order that \bar{S}' arise from \bar{S} by a transformation A , it is necessary that equations (101) hold when ξ is replaced by x . Evidently this is impossible.

9. THEOREMS OF PERMUTABILITY FOR TRANSFORMATIONS K_m OF SURFACES C AND FOR TRANSFORMATIONS A_m OF SURFACES Ω

By means of the results of the first part of § 7 we prove

THEOREM VIII. *If surfaces C_1 and C_2 are obtained from a surface C by transformations K_{m_1} and K_{m_2} , a fourth surface C_{12} can be found without quadratures which is in the relation of transformations K'_{m_1} and K'_{m_2} with C_1 and C_2 .*

The coördinates of a surface S_{12} forming a quatern with C , C_1 , C_2 are given by equations of the form (78), where θ_{21} must satisfy equations (72) with $i = 2, j = 1$, and the function λ_{21} is given by (76) with $i = 2, j = 1$. Since t, t_1 , and t_2 are solutions of the point equations of C, C_1 , and C_2 respectively, the function t_{12} , given by

$$(102) \quad \lambda_{21} \theta_2 t_{12} = \theta_{21} \lambda_2 t - \lambda_2 \theta_1 t_2 + \lambda_1 \theta_2 t_1,$$

* Journal de mathématiques pures et appliquées, ser. 4, vol. 7 (1891), p. 228.

is a solution of the point equation of S_{12} , as follows from (78). The point equations of C , C_1 and C_2 admit also the respective solutions $\sum x^2 - t^2$, $\sum x_1^2 - t_1^2$, and $\sum x_2^2 - t_2^2$. If the point equation of S_{12} admits the solution $\sum x_{12}^2 - t_{12}^2$, S_{12} is a surface C . This will be the case, if

$$(103) \quad \lambda_{21} \theta_2 (\sum x_{12}^2 - t_{12}^2) = \theta_{21} \lambda_2 (\sum x^2 - t^2) - \lambda_2 \theta_1 (\sum x_2^2 - t_2^2) \\ + \lambda_1 \theta_2 (\sum x_1^2 - t_1^2).$$

This is reducible by means of (102) to

$$\theta_{21} \{ \lambda_2 \theta_1 [(x_2 - x)^2 - (t_2 - t)^2] - \lambda_1 \theta_2 [\sum (x_1 - x)^2 - (t_1 - t)^2] \\ + \lambda_1 \theta_1 \theta_2 [\sum (x_1 - x_2)^2 - (t_1 - t_2)^2] \} = 0.$$

With the aid of equations of the type (9), (63), (64), and (92) for the surfaces C_1 and C_2 this equation can be put in the form

$$(104) \quad \lambda_2 \theta_{21} (m_2 - m_1) = m_2 \lambda_2 \theta_1 + m_1 \lambda_1 \theta_2 - \Phi_{12} + m_1 m_2 \lambda_1 \lambda_2 (t_1 - t) (t_2 - t).$$

Making use of equations analogous to (64) and (94), we can show that θ_{21} so defined satisfies equations (72) for $i = 2$, $j = 1$. Consequently S_{12} is a surface C and it is determined without quadratures.

Suppose now that we have a quatern (C, C_1, C_2, C_{12}) and let C_0, C_{10}, C_{20} be surfaces conjugate to C, C_1, C_2 respectively. We propose to show that the surface C' which is the transform of C_{12}, C_{10} , and C_{20} in accordance with Theorem V is conjugate to C_{12} .

From (83) it follows that the coördinates x', y', z' of C' are given by equations of the form

$$x' (\theta_{12} \lambda_0 + \theta_{20} \lambda_2 - \theta_{10} \lambda_2) = \theta_{20} \lambda_2 x_1 - \theta_{10} \lambda_2 x_2 + \theta_{12} \lambda_0 x_0.$$

With the aid of (78) this may be given the form

$$(\theta_{12} \lambda_0 + \theta_{20} \lambda_2 - \theta_{10} \lambda_2) (x' - x_{12}) = (x - x_1) (AB + \lambda_1 \theta_2 B - \lambda_2 \theta_2 C) \\ + (x_2 - x) \frac{\lambda_2}{\lambda_1} (AC + \lambda_1 \theta_1 B - \lambda_2 \theta_1 C) + (x - x_0) A (A + \lambda_1 \theta_2 - \lambda_2 \theta_1),$$

where

$$A = \lambda_1 \theta_{12} - \lambda_1 \theta_2 + \lambda_2 \theta_1, \quad B = \theta_{20} \lambda_2 + \theta_2 \lambda_0, \quad C = \theta_{10} \lambda_1 + \theta_1 \lambda_0.$$

This equation is satisfied also by the y 's, z 's, and t 's.

From equations analogous to (9), (63), (64), (67), and (104) we have

$$\sum (x_i - x)^2 - (t_i - t)^2 = 2\theta_i / \lambda_i m_i \quad (i = 1, 2), \\ \sum (x_0 - x)^2 - (t_0 - t)^2 = 0, \\ \sum (x_1 - x) (x_2 - x) - (t_1 - t) (t_2 - t) = \frac{\Phi_{12}}{m_1 m_2 \lambda_1 \lambda_2} - (t_1 - t) (t_2 - t) \\ = \frac{1}{m_1 m_2 \lambda_1 \lambda_2} [A (m_2 - m_1) + m_1 \lambda_2 \theta_1 + m_2 \lambda_1 \theta_2],$$

$$\sum (x_1 - x)(x_0 - x) - (t_1 - t)(t_0 - t) = C/m_1 \lambda_1 \lambda_0,$$

$$\sum (x_2 - x)(x_0 - x) - (t_2 - t)(t_0 - t) = B/m_2 \lambda_2 \lambda_0.$$

With the aid of these equations we show that

$$\sum (x' - x_{12})^2 - (t' - t_{12})^2 = 0.$$

Hence (§ 6) C' and C_{12} are conjugate surfaces and from Theorem VI we have

THEOREM IX. *If \bar{S}_1 and \bar{S}_2 are two surfaces Ω obtained from \bar{S} , a given surface Ω , by transformations A_{m_1} and A_{m_2} , there can be found without quadratures a fourth surface Ω which is in the relation of transformations A'_{m_1} and A'_{m_2} with both \bar{S}_1 and \bar{S}_2 .*

We have seen that when a surface C is known, so also are two surfaces Ω , namely the sheets \bar{S} and \bar{S}' of the envelope of spheres of radius t and centers on C . In § 5 we showed that as soon as the surface C_0 has been found, the surface C'_0 follows at once. Since the relations between C and C_0 , and C and C'_0 are of the same kind, the theorem of permutability gives a surface C'_{10} such that the surfaces C , C'_0 , C_1 , C'_{10} are related among themselves in the same manner as C , C_0 , C_1 , C_{10} . Consequently when a transformation A_{m_1} of a surface \bar{S} into a surface \bar{S}_1 is known, there follows a transformation A'_{m_1} from \bar{S}' into a surface \bar{S}'_1 , and the latter transformation can be found without quadratures. Moreover, \bar{S}_1 and \bar{S}'_1 envelop a two-parameter family of spheres whose centers are on C_1 and whose radii are given by t_1 . Hence the four surfaces \bar{S} , \bar{S}' , \bar{S}_1 , \bar{S}'_1 form a quatern under a theorem of permutability for general transformations of Ribaucour.*

There is a surface \bar{S}'' which with \bar{S} constitutes the envelope of the two-parameter family of spheres whose centers are on C_0 and whose radii are given by t_0 . In this way we get a quatern \bar{S} , \bar{S}'' , \bar{S}_1 , \bar{S}'_1 similar to the last one mentioned. In a similar manner other quaterns for general transformations of Ribaucour can be found.

10. PARALLEL SURFACES Ω

In our general discussion of transformations K we pointed out the fact† that when we take $\theta_1 = -\lambda_1 = 1$, the surface S_1 is an associate of S . In this case equations (3) and (61) reduce to

$$(105) \quad \begin{aligned} \frac{\partial x_1}{\partial u} &= -\rho \frac{\partial x}{\partial u}, & \frac{\partial x_1}{\partial v} &= \rho \frac{\partial x}{\partial v}, \\ \frac{\partial t_1}{\partial u} &= -\rho \frac{\partial t}{\partial u}, & \frac{\partial t_1}{\partial v} &= \rho \frac{\partial t}{\partial v}, \end{aligned}$$

* Bianchi was the first to show that there is a theorem of permutability of transformations of Ribaucour.

† L. c., p. 401.

and x_1, y_1, z_1, t_1 are solutions of the equation

$$(106) \quad \frac{\partial^2 \phi}{\partial u \partial v} = \frac{\partial \log \sqrt{\rho}}{\partial v} \frac{\partial \phi}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u} \frac{\partial \phi}{\partial v}.$$

In consequence of (105) and similar equations in y_1 and z_1 we have

$$\begin{aligned} \frac{\partial}{\partial u} (x_1^2 + y_1^2 + z_1^2 - t_1^2) &= -\rho \left(x_1 \frac{\partial x}{\partial u} + y_1 \frac{\partial y}{\partial u} + z_1 \frac{\partial z}{\partial u} - t_1 \frac{\partial t}{\partial u} \right), \\ \frac{\partial}{\partial v} (x_1^2 + y_1^2 + z_1^2 - t_1^2) &= \rho \left(x_1 \frac{\partial x}{\partial v} + y_1 \frac{\partial y}{\partial v} + z_1 \frac{\partial z}{\partial v} - t_1 \frac{\partial t}{\partial v} \right). \end{aligned}$$

Making use of these results and (36), we can show that $x_1^2 + y_1^2 + z_1^2 - t_1^2$ is a solution of (106). Hence the associate surface of a surface C is also a surface C , and consequently determines a surface Ω . It is our purpose now to discover the relation between the latter surface Ω and the one determined by the given surface C .

If we put $\theta_1 = -\lambda_1 = 1$ in equations (72) with $i = 1, j = 0$, we find with the aid of (70), (73), and (76)

$$(107) \quad \theta_{10} = \lambda_0, \quad \theta_{01} = 1, \quad \lambda_{10} = \theta_0, \quad \lambda_{01} = -1.$$

And the equation analogous to (78) reduces to

$$(108) \quad \lambda_0 (x_0 - x) = \theta_0 (x_{10} - x_1).$$

From the values for θ_{01} and λ_{01} , it follows that S_0 and S_{10} are associate surfaces. Hence if S_0 is a surface C so also is S_{10} . Suppose now that we have four surfaces C, C_1, C_0, C_{10} forming a quatern such that the pairs C and C_1, C_0 and C_{10} are associate. We wish to show that if C_0 is a conjugate of C so likewise is C_{10} of C_1 .

From (108) it follows that the auxiliary function t_{10} of C_{10} is given by

$$(109) \quad \lambda_0 (t_0 - t) = \theta_0 (t_{10} - t_1).$$

The condition, analogous to (67), that C_{10} be a conjugate of C_1 becomes, in consequence of (107),

$$(110) \quad t_{10} = t_1 - T_{10}/\theta_0 m_0.$$

Hence from (67), (109), and (110) we have

$$(111) \quad T_{10} = T_0.$$

In consequence of (105) and (107) equations (96) are reducible to

$$(112) \quad \frac{\partial T_{10}}{\partial u} = m_0 (\lambda_0 - \rho \theta_0) \frac{\partial t}{\partial u}, \quad \frac{\partial T_{10}}{\partial v} = m_0 (\lambda_0 + \rho \theta_0) \frac{\partial t}{\partial v}.$$

When we substitute the value of T_{10} given by (111), the resulting equations

are the same as the first two of (43). Hence all the conditions are satisfied and C_1 and C_{10} are conjugate. Accordingly we have

THEOREM X. *If four surfaces C, C_1, C_0, C_{10} form a quatern such that C and C_1 are associate and likewise C_0 and C_{10} , and if C_0 is a conjugate of C , so also is C_{10} a conjugate of C_1 . Moreover, the surfaces Ω normal respectively to the lines joining corresponding points on C and C_0 and on C_1 and C_{10} correspond with parallelism of tangent planes.*

The last part of this theorem follows from (108). Hence when we take $\theta_1 = 1$, the transform of \bar{S} and \bar{S} itself correspond with parallelism of tangent planes instead of constituting the envelope of a two-parameter family of spheres.

From the foregoing results and those of § 9 follows

THEOREM XI. *If $\bar{S}, \bar{S}_1, \bar{S}_2, \bar{S}'$ are four surfaces forming a quatern in accordance with the theorem of permutability of transformations A_m and if \bar{S} and \bar{S}_1 correspond with parallelism of tangent planes, so also do \bar{S}_2 and \bar{S}' .*

11. ISOTHERMIC SURFACES

An isothermic surface is characterized by the property that when its lines of curvature are parametric, the fundamental quantities E and G are equal, or can be made so by a change of the parameters. If we put

$$\sqrt{E} = \sqrt{G} = e^\omega,$$

equations (24) become

$$(113) \quad \frac{\partial}{\partial v} \left(\frac{1}{\rho_1} \right) = \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \frac{\partial \omega}{\partial v}, \quad \frac{\partial}{\partial u} \left(\frac{1}{\rho_2} \right) = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \frac{\partial \omega}{\partial u}.$$

From these results and (34) it follows that an isothermic surface is a surface Ω .

In this case \bar{S} itself is one of the two surfaces C described by points on the normal, since the point equation of \bar{S} has equal invariants and admits the solution $x^2 + y^2 + z^2$. If we call it C , we have $t = 0$. From (42) it follows at once that a_0 and b_0 are zero and from equations analogous to the last two of (10) that w_0 is constant.

Now $\rho = e^{-2\omega}$, and consequently from (43) we have

$$(114) \quad \theta_0 = \frac{w_0}{2m_0} e^{2\omega} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right), \quad \lambda_0 = \frac{w_0}{2m_0} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right).$$

If \bar{S}_1 also is to be isothermic we must have $t_1 = 0$. Equations (67) and (95) reduce to

$$t_0 = -w_0/\lambda_0 m_0, \quad t_{10} = -w_{10}/\lambda_{10} m_0,$$

where evidently w_{10} is a constant. In order that these values may satisfy (89), we must have $w_{10} = w_0$.

For the present case equations (2) are

$$(115) \quad \frac{\partial \lambda_0}{\partial u} = -e^{-2\omega} \frac{\partial \theta_0}{\partial u}, \quad \frac{\partial \lambda_0}{\partial v} = e^{-2\omega} \frac{\partial \theta_0}{\partial v}.$$

With the aid of (113) it is readily shown that the functions (114) satisfy equations (115). Darboux* pointed out that this value θ_0 is a solution of the point equation of isothermic surfaces without an indication of its bearing in the present case.

We have applied† the general results of § 1 to the case where S is isothermic and will not repeat any of that investigation. We content ourselves with the observation that the point M_0 harmonic to a point M of an isothermic surface S with respect to the corresponding centers of principal curvature of S generates a surface S_0 upon which the developables of the congruence of normals to S cut out a conjugate system with equal point invariants. Moreover, the transformations D_m of S determine transformations K_m of S_0 .

12. SURFACES WITH ISOTHERMAL SPHERICAL REPRESENTATION OF THEIR LINES OF CURVATURE

There is one class of surfaces Ω whose transformations cannot be handled as in the preceding pages. They are surfaces whose lines of curvature are represented on the gaussian unit sphere by an isothermal system. The normals to such a surface form a congruence of Ribaucour,‡ and the only normal congruence of Ribaucour. The developables of the congruence meet the middle surface of the congruence in a conjugate system with equal point invariants. Hence the middle surface is a surface C and the surface C_0 is at infinity.

Let M be any minimal surface and S a surface corresponding to M with orthogonality of linear elements; for the sake of brevity we say that S is an *ortho-surface* of M . If lines be drawn through points of S parallel to the corresponding normals to M , these lines form a normal congruence of Ribaucour and S is the middle surface. Moreover, this is the most general way of obtaining normal congruences of Ribaucour.§ We consider this case and denote by Σ a surface normal to such a normal congruence of Ribaucour.

Elsewhere|| we have established transformations of the Ribaucour type of surfaces Σ which are such that if Σ and Σ_1 are two surfaces in the relation of the transformation there considered, and M and M_1 are the minimal surfaces

* Annales de l'école normale supérieure, sér. 3, vol. 16 (1899), p. 503.

† M., pp. 422-428.

‡ E., p. 422.

§ E., pp. 420-422.

|| These Transactions, vol. 9 (1908), pp. 149-177. Hereafter a reference to this memoir will be of the form M_1 , p. —.

associate to Σ and Σ_1 respectively, then M and M_1 are the focal surfaces of a W -congruence. It is our purpose to apply to this case the general results of M ., §§ 6, 7, in order to show that the middle surfaces of the congruences of normals to Σ and Σ_1 are surfaces C in the relation of a transformation K_m . In that event the relation between Σ and Σ_1 is essentially that of a transformation A_m .

Consider a minimal surface M referred to its asymptotic lines. Its linear element can be given the form

$$(116) \quad ds'^2 = e^{-2\psi} (du^2 + dv^2),$$

where ψ is a solution of the equation

$$(117) \quad \frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = e^{-2\psi}.*$$

The direction-cosines, X, Y, Z , of the normal to M are solutions of the equation

$$(118) \quad \frac{\partial^2 w}{\partial u \partial v} + \frac{\partial \psi}{\partial v} \frac{\partial w}{\partial u} + \frac{\partial \psi}{\partial u} \frac{\partial w}{\partial v} = 0.$$

Each solution of this equation determines an ortho-surface of M . In fact, if w_2 is such a solution, here assumed to be linearly independent of X, Y , and Z , the coördinates of the corresponding ortho-surface S are given by the quadratures†

$$(119) \quad \frac{\partial x}{\partial u} = e^{2\psi} \left(w_2 \frac{\partial X}{\partial u} - X \frac{\partial w_2}{\partial u} \right), \quad \frac{\partial x}{\partial v} = -e^{2\psi} \left(w_2 \frac{\partial X}{\partial v} - X \frac{\partial w_2}{\partial v} \right).$$

It is readily found that the point equation of S is of the form (1) where now

$$(120) \quad 1/\sqrt{\rho} = e^\psi w_2.$$

Since w_2 is a solution of (118), there exists a function ϕ_2 defined by

$$(121) \quad \frac{\partial \phi_2}{\partial u} = e^{2\psi} \frac{\partial w_2}{\partial u}, \quad \frac{\partial \phi_2}{\partial v} = -e^{2\psi} \frac{\partial w_2}{\partial v}.$$

Moreover, ϕ_2 is a solution of (1). And it is easily shown that E, F, G for S have the forms (35) and (36) when we take $t = \phi_2$. Hence S is a surface C .

If $X_1, Y_1, Z_1; X_2, Y_2, Z_2$ denote the direction-cosines of the tangents to the parametric curves of the spherical representation of M , we have‡

$$\begin{aligned} \frac{\partial X}{\partial u} &= e^{-\psi} X_1, & \frac{\partial X_1}{\partial u} &= -\frac{\partial \psi}{\partial v} X_2 - e^{-\psi} X, & \frac{\partial X_2}{\partial u} &= \frac{\partial \psi}{\partial v} X_1, \\ \frac{\partial X}{\partial v} &= -e^{-\psi} X_2, & \frac{\partial X_1}{\partial v} &= \frac{\partial \psi}{\partial u} X_2, & \frac{\partial X_2}{\partial v} &= -\frac{\partial \psi}{\partial u} X_1 + e^{-\psi} X. \end{aligned}$$

* M_1 , p. 151.

† M_1 , p. 152.

‡ M_1 , p. 151.

Hence equations (119) may be written

$$(122) \quad \frac{\partial x}{\partial u} = e^\psi w_2 X_1 - \frac{\partial \phi_2}{\partial u} X, \quad \frac{\partial x}{\partial v} = e^\psi w_2 X_2 - \frac{\partial \phi_2}{\partial v} X.$$

From these expressions we find that the direction-cosines of the normal to S , say X_0, Y_0, Z_0 , are given by equations of the form

$$X_0 = \frac{e^\psi w_2}{H} \left(e^\psi w_2 X + \frac{\partial \phi_2}{\partial u} X_1 + \frac{\partial \phi_2}{\partial v} X_2 \right),$$

where

$$H^2 = EG - F^2 = e^{2\psi} w_2^2 \left[e^{2\psi} w_2^2 + \left(\frac{\partial \phi_2}{\partial u} \right)^2 + \left(\frac{\partial \phi_2}{\partial v} \right)^2 \right].$$

From (122) with the aid of the above equations we find for the second fundamental coefficients, D, D'' , of S the expressions

$$D = \sum X_0 \frac{\partial^2 x}{\partial u^2} = - \frac{e^{2\psi} w_2^2}{H} \left[\frac{\partial^2 \phi_2}{\partial u^2} + w_2 + \frac{\partial \phi_2}{\partial v} \frac{\partial \psi}{\partial v} - \frac{\partial \phi_2}{\partial u} \frac{\partial \psi}{\partial u} \right],$$

$$D'' = \sum X_0 \frac{\partial^2 x}{\partial v^2} = - \frac{e^{2\psi} w_2^2}{H} \left[\frac{\partial^2 \phi_2}{\partial v^2} - w_2 - \frac{\partial \phi_2}{\partial v} \frac{\partial \psi}{\partial v} + \frac{\partial \phi_2}{\partial u} \frac{\partial \psi}{\partial u} \right].$$

In consequence of these values and (120) equations (47) reduce to

$$T_0 e^{-2\psi} = m_0 w_2 (\rho \theta_0 - \lambda_0), \quad T_0 e^{-2\psi} = m_0 w_0 (\rho \theta_0 + \lambda_0).$$

Also from the above values and (48) we have

$$w_0 = 1/H\rho w_2, \quad T_0 = 1/w_2,$$

so that the preceding equations are equivalent to

$$\lambda_0 = 0, \quad \theta_0 = 1/m_0.$$

Hence, as seen from equations analogous to (9), C_0 is at infinity. Consequently the general formulas hold in this case also.

In accordance with M., § 7 we seek a quatern of W -congruences for which the two focal surfaces M and M_1 are minimal. From the general theory of associate surfaces it follows that each surface associate to a given surface, say S_0 , determines a W -congruence for which S_0 is one of the focal surfaces. We have shown* that there exist solutions of equation (118) such that if w_1 is such a solution and ϕ_1 the function given by

$$(123) \quad \frac{\partial \phi_1}{\partial u} = e^{2\psi} \frac{\partial w_1}{\partial u}, \quad \frac{\partial \phi_1}{\partial v} = -e^{2\psi} \frac{\partial w_1}{\partial v},$$

these functions satisfy the equations

* M₁, p. 155.

$$\begin{aligned}
 \frac{\partial^2 \phi}{\partial u^2} &= \frac{\partial \psi}{\partial u} \frac{\partial \phi}{\partial u} - \frac{\partial \psi}{\partial v} \frac{\partial \phi}{\partial v} + me^{2\psi} w + m\phi - w, \\
 \frac{\partial^2 \phi}{\partial u \partial v} &= \frac{\partial \psi}{\partial v} \frac{\partial \phi}{\partial u} + \frac{\partial \psi}{\partial u} \frac{\partial \phi}{\partial v}, \\
 \frac{\partial^2 \phi}{\partial v^2} &= -\frac{\partial \psi}{\partial u} \frac{\partial \phi}{\partial u} + \frac{\partial \psi}{\partial v} \frac{\partial \phi}{\partial v} + me^{2\psi} w - m\phi + w, \\
 e^{-2\psi} \left[\left(\frac{\partial \phi}{\partial u} \right)^2 + \left(\frac{\partial \phi}{\partial v} \right)^2 \right] &= 2m\phi w - w^2,
 \end{aligned}
 \tag{124}$$

where m is a constant; moreover, the associate of M given by w_1 determines a W -congruence for which the second focal surface is minimal. We assume that M_1 is a minimal surface so determined. The linear element of its spherical representation is of the form (116) where now

$$e^{\psi_1} = e^{-\psi} \phi_1 / w_1. * \tag{125}$$

Since w_2 is a solution of (118) it determines an associate of M which in turn defines a W -congruence of which M is one focal surface; we call the second focal surface Σ_2 , and denote by Σ' the fourth surface such that M , M_1 , Σ_2 , and Σ' are the focal surfaces of a quatern of W -congruences. In order to find Σ' we note that if we put

$$\sigma = e^{\psi} w \tag{126}$$

equation (118) is replaced by

$$\frac{\partial^2 \sigma}{\partial u \partial v} = e^{-\psi} \frac{\partial^2 e^{\psi}}{\partial u \partial v} \sigma. \tag{127}$$

Hence when w in (126) is replaced by w_1 and w_2 we have the values of σ , namely σ_1 and σ_2 , which determine M_1 and Σ_2 . The corresponding equations for M_1 and Σ_2 are of the form†

$$\frac{\partial^2 \sigma'_i}{\partial u \partial v} = \sigma_i \frac{\partial^2}{\partial u \partial v} \left(\frac{1}{\sigma_i} \right) \cdot \sigma'_i \quad (i = 1, 2). \tag{128}$$

Moreover, the respective functions σ'_1 and σ'_2 which determine Σ' as the second focal sheet of the W -congruences with M_1 and Σ_2 as the first focal sheets respectively are given by M. (92), namely

$$\begin{aligned}
 \frac{\partial}{\partial u} (\sigma_i \sigma'_i) &= -\sigma_i^2 \frac{\partial}{\partial u} \left(\frac{\sigma_j}{\sigma_i} \right), & \frac{\partial}{\partial v} (\sigma_i \sigma'_i) &= \sigma_i^2 \frac{\partial}{\partial v} \left(\frac{\sigma_j}{\sigma_i} \right) \\
 & & (i = 1, 2, j = 1, 2; i \neq j).
 \end{aligned}
 \tag{129}$$

If w'_1 denotes the function determining the associate surface of M_1 , which

* M₁, p. 157.

† M. (91).

in turn determines the W -congruence whose focal surfaces are M_1 and Σ' , then the equation analogous to (126) is

$$(130) \quad \sigma'_1 = e^{\psi_1} w'_1.$$

In consequence of (125) we have

$$(131) \quad \sigma_1 \sigma'_1 = \phi_1 w'_1,$$

so that equations (129) may be replaced by

$$(132) \quad \frac{\partial}{\partial u}(\phi_1 w'_1) = -e^{2\psi} w_1^2 \frac{\partial}{\partial u} \left(\frac{w_2}{w_1} \right), \quad \frac{\partial}{\partial v}(\phi_1 w'_1) = e^{2\psi} w_1^2 \frac{\partial}{\partial v} \left(\frac{w_2}{w_1} \right).$$

Just as w_2 determines the ortho-surface S of M , so likewise w'_1 determines an ortho-surface S_1 of M_1 . In M., § 7, we showed that S and S_1 are in the relation of a transformation K . It is our purpose now to find the equations of this transformation.

If we put

$$(133) \quad \alpha = Xe^\psi, \quad \beta = Ye^\psi, \quad \gamma = Ze^\psi,$$

where X, Y, Z are the direction-cosines of the normal to M , it follows from (126) that α, β, γ are solutions of (127).

From M. (84) and the expression for ρ_1 analogous to (120), it follows that the coördinates x_1, y_1, z_1 of S_1 are of the form

$$x_1 = \alpha' / \sqrt{\rho_1} = \alpha' e^{\psi_1} w'_1 = \alpha' \sigma'_1,$$

where α', β', γ' , bear to the direction-cosines of the normal to Σ' a relation analogous to that expressed by (133). If α_1 and α_2 denote similar functions for M_1 and Σ_2 respectively, we have from M. (81) and equations analogous to (133)

$$\alpha_2 = \sqrt{\rho} x = e^{-\psi} x / w_2, \quad \alpha_1 = X' e^{\psi_1} = X' e^{-\psi} \phi_1 / w_1,$$

where X', Y', Z' denote the direction-cosines of M_1 .

Between these functions there exists the relation

$$\alpha' = \alpha - \frac{\sigma_2}{\sigma_1} (\alpha_1 - \alpha_2),$$

as follows from M. (96). When the above values are substituted, we obtain

$$(134) \quad x_1 + \sigma_2 e^{\psi_1} X' = x + \sigma'_1 e^\psi X.$$

Since similar equations hold in the y 's and z 's, we have that lines through corresponding points of S and S_1 parallel to the corresponding normals to M and M_1 meet in a point. We note that as thus drawn these lines generate normal congruences of Ribaucour.

If r denotes the distance from a point of S to the corresponding point on one of the surfaces orthogonal to the first of these congruences, we must have

$$\sum X \frac{\partial}{\partial u} (x + rX) = 0, \quad \sum X \frac{\partial}{\partial v} (x + rX) = 0.$$

With the aid of (119) we find that $r = \phi_2 = t$, which shows that, as in the general case, the auxiliary function t measures the distance to the orthogonal surface.

The corresponding function t_1 for S_1 must be ϕ'_1 where

$$(135) \quad \frac{\partial \phi'_1}{\partial u} = e^{2\psi_1} \frac{\partial w'_1}{\partial u}, \quad \frac{\partial \phi'_1}{\partial v} = -e^{2\psi_1} \frac{\partial w'_1}{\partial v}.$$

We have now to show that these values of t and t_1 satisfy equations (61). In the first place we must find the expressions for θ and λ in terms of the functions used in this section.

From M. (85), M. (94), (120), and (131) we have

$$\theta = -\phi_1 w'_1.$$

From (5) and the expression for ρ_1 analogous to (120), namely $\sqrt{\rho_1} = e^{-\psi_1}/w'_1$ we find

$$\lambda = w_1/w_2.$$

Substituting these expressions in (61), we find that both are satisfied provided that

$$(136) \quad \phi'_1 = \phi_2 + (w'_1 - w_2) \phi_1/w_1.$$

It is readily shown that this value of ϕ'_1 satisfies equations (135).

In consequence of (126) and (130) equation (136) can be written

$$(137) \quad \phi'_1 + \sigma_2 e^{\psi_1} = \phi_2 + \sigma'_1 e^{\psi}.$$

From this it follows that the point whose coördinates are of the form (134) is equidistant from the two surfaces normal to the two congruences of Ribaucour just considered, and whose cartesian coördinates are of the form

$$(138) \quad x + \phi_2 X, \quad x_1 + \phi'_1 X',$$

respectively.

The functions ϕ_1 and w_1 satisfying equations (123) and (124) determine the most general minimal surface M_1 such that M and M_1 are the focal surfaces of a W -congruence. From the general theory* we know that w_1 determines an ortho-surface, whose coördinates are given by quadratures similar to (119), and that this surface is associate to a unique ortho-surface of M_1 , in which

* E., pp. 417-420.

sense it may be said to determine M_1 . In view of these various results we have

THEOREM XIV. *If two minimal surfaces, M and M_1 , are the focal surfaces of a W -congruence, and S is any ortho-surface of M other than the one determining M_1 , there exists an ortho-surface S_1 of M_1 , which can be found without quadratures, such that S and S_1 are two surfaces C in the relation of a transformation K . Moreover, the lines through corresponding points on S and S_1 parallel to the normals to M and M_1 respectively at corresponding points meet in points P and are normal to two surfaces Ω which envelop a two-parameter family of spheres whose centers are the points P .*

Consequently surfaces with isothermal spherical representation of their lines of curvature admit transformations A_m .

PRINCETON UNIVERSITY,
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